SPHERICAL TRIGONOMETRY

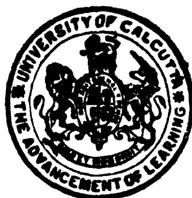
TEXT BOOK

OF

SPHERICAL TRIGONOMETRY

BY

PRAMATHA NATH MITRA, M.A.
LECTURER IN PURE MATHEMATICS IN THE UNIVERSITY OF CALCUTTA
SOMETIME SECRETARY OF THE CALCUTTA MATHEMATICAL SOCIETY



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To

THE LOVING MEMORY

OF

MY REVERED FATHER

BARODA PRASANNA MITRA

PREFACE

The present work has evolved out of the lectures delivered to the Post-Graduate students of the University of Calcutta. It is intended as an introductory text-book on Spherical Trigonometry and an attempt has been made to present the subject-matter in as simple a manner as possible. The book has been brought to the standard required for the examinations of Indian Universities. It contains all the propositions which a student has and ought to learn to have a fairly comprehensive knowledge of the Trigonometry of Spheres, and thus it paves the way for higher study in Spherical Astronomy.

As the book consists mainly of formulae and the applications thereof, a large number of examples has been appended for solution by the students.

A short historical introduction has been given at the beginning, showing the successive stages of the development of the subject. It arose out of the growing need for the study of the heavens. It is interesting to note that the fundamental formulae were all known to Hindu Astronomers thousands of years ago and are of Indian origin, but owing to their conservative spirit, any record of their work is wholly wanting. It was *Sūrya Siddhānta* which brought to light the achievement of Indian mathematicians,

and this was followed by several works on the subject, showing thereby that the ancient Hindus were far advanced in Astronomy. In the body of the book reference to authors of the respective theorems has in most cases been given.

In the preparation of this book I had to consult the existing treatises and several memoirs on the subject, and my thanks are due to their respective authors. For the history of the subject, among other works, I was greatly influenced by the monumental works of Dr. D. E. Smith and the late Dr. F. Cajori and my thanks are due to them. I am also indebted to Dr. S. M. Ganguli, D.Sc., P.R.S., Lecturer in Higher Geometry in the University of Calcutta, for his valuable suggestions.

I have also to express my thanks to the authorities of the University of Calcutta for their consent to publish the book, and to the officers and the staff of the University Press, for the pains they have taken in the printing of the book.

In conclusion, I hope that the present book will tend a little towards the advancement of Mathematical learning of our students ; it is for them that the book has been written and it is in their profit that I shall look for my reward.

UNIVERSITY OF CALCUTTA :
July, 1935.

P. N. MITRA.

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HISTORICAL INTRODUCTION

The early history of the Science of Spherical Trigonometry is veiled in obscurity. In the prehistoric age, primitive men attributed every physical phenomenon to the agency of some Superhuman Being. To them the secret of the stars was closely connected with the secrets of human destiny. It was this that led the Hindus of India and the Babylonian shepherd to observe the stars and to speculate on their meaning. Thus developed the folklore in India as also in the temples along the Nile and in Mesopotamia. As years advanced, observations of the heavens increased, which led to the measurement of angles, and thus the science of Astronomy had its beginning. The ancient Hindus however left no authentic record of their mathematical achievement. They were very conservative and would hardly impart their knowledge to their friends and disciples. Moreover they had little sympathy with those outside their own caste. It is only in some special case that a favourite disciple could acquire the knowledge and learning of his teacher. With the passing away of a master mind, all his mathematical achievements were lost in oblivion. There is sufficient evidence to show that schools existed very early in India, where mathematics was looked upon as a very important

branch of learning, but for the reasons aforesaid a general literature on the subject is wholly lacking. All that we can learn of them are gathered from the two great epics, the Mahabharata and the Ramayana, the Vedas and other ancient literatures which show that the Hindus from ancient times paid considerable attention to astronomy.* The oldest astronomical instrument dates as early as 1800 B.C.

The study of scientific astronomy began in Greece with **Thales** (640—546 B.C.). He succeeded in predicting a solar eclipse which occurred on the 28th May, 585 B.C. **Pythagoras** (580–500 B.C.) asserted that Earth was spherical in shape. His teachings reveal much more of Indian than of the Greek civilisation in which he was born. It was left for **Parmenides** of Elea (460 B.C.) to teach at Athens the doctrine of the sphericity of the Earth. **Eudoxus** of Cindus (408—355 B.C.) is said to have introduced the study of spherics (mathematical astronomy) in Greece. **Euclid** of Alexandria (fl. 300 B.C.) wrote a book called *Phacnomena* dealing with the celestial sphere. **Eratosthenes** of Alexandria (274—194 B.C.) took the noteworthy step in geodesy by his measurement of the circumference and diameter of the Earth. He also found the obliquity of the ecliptic

* **G. Oppert,** *On the Original Inhabitants of Bharatavarsa or India*, **London,** 1893.

R. C. Dutt, *A History of Civilisation in Ancient India*, London, 1893.

to be 23°51′ 20″. **Archimedes** of Syracuse (287—212 B.C.) devoted a portion of his work on sphere. As yet we have got nothing which can be called trigonometrical.

Hipparchus of Nicæa (180—125 B.C.) wrote a famous work on astronomy, in which he needed to measure angles and distances on a sphere, and hence he developed a kind of Spherical Trigonometry. He also worked out a table of chords, *i. e.*, of double sines of half the angle, and thus was begun the science of Trigonometry. **Menelaus** of Alexandria (fl. 100 A.D.) wrote a treatise on sphere *Sphaericorum Libri III* dealing with geometrical properties of spherical triangles. His proposition *Regula sex quantitatum* is well known. He also wrote six books on the calculation of chords. The interest in astronomy had induced more progress in spherical rather than in plane trigonometry. **Claudius Ptolemaeus** (85—165 A.D.) brought together in his great work, *Almagest* in 13 books, the discoveries of his predecessors. He devoted chapters of his first book to trigonometry and spherical trigonometry. He elaborated the table of sines already used by Hipparchus. He created, for astronomical use, a *trigonometry* remarkably perfect in form. **Pappus** of Alexandria (fl. 300 A.D.) devoted his sixth book in *Mathematical Collections* to the treatment of sphere.

From 2000 B.C. down to 300 B.C. we have no record of Indian astronomy save the glimpses we

have from the Vedic writings. The Vedic literatures were probably written about 2500—1500 B.C., though composed much earlier; the Vedangas were written several centuries later. The ritualistic rules of the *Sulvasutras* were composed about 500 B.C. The Hindus were in the habit of putting into verse all mathematical results they obtained, and of clothing them in obscure and mystic language, which though well adapted to aid the memory of him who already understood the subject, was often unintelligible to the uninitiated. From the period of invasion of India by **Alexander the Great** in 327 B.C., there was regular intercourse between the Hindu and Greek mathematicians, which influenced their respective astronomies to a certain extent. Before the beginning of the Christian era, there were numerous invasions from the North which seriously interfered with the spread of Greek science, and in the fourth century A.D., with the appearance of *Surya Siddhanta*—the first important work on Astronomy in India—we find the astronomy of Greece replaced by the Astronomy of Hindus. The mathematical formulae of *Sulvasutras* now gave place to the mathematics of stars. Spherical trigonometry and astronomy were treated scientifically by **Aryabhatta** (475—550 A.D.) in his *Aryabhatiyam* and *Gola*. Next comes **Varahamihira** * (505—587 A.D.) whose work *Pañca Siddhāntika* shows

* According to some tradition **Varaha** and **Mihira** are two different persons—father and son.

an advanced state of mathematical astronomy. He describes the five Siddhantas which had been written before his time but places the *Sūryå Siddhānta* at the head. Among the five is the *Paulisa Siddhānta* which contains an excellent summary of early Hindu Trigonometry. Varahamihira taught the sphericity of the earth. The most prominent of the Hindu mathematicians, of the seventh century, was **Brahmagupta,** who was born in 598 A.D. He wrote his astronomical works *Brāhma-sphuta-siddhānta* in 628 A.D. and *Khandakhādyaka* in 665 A.D. It was he who taught the Arabs astronomy long before they became acquainted with Ptolemy's work. The famous *Sindhind* and *Alarkand* of the Arabs are the translations of the two books of Brahmagupta. The *cosine and sine* theorems for oblique-angled spherical triangles are implied in the rules of Varahamihira and Brahmagupta. The triadic relations for right-angled spherical triangles were known to the Hindu mathematicians and were used by them to solve spherical triangles. In the reign of **Caliph Almansur** of Bagdad a Hindu Astronomer named **Kankah** went to his court with astronomical tables * in 766 A. D., which were translated into Arabic. Thus Hindu mathematics

* It is generally believed that this was the *Brāhma-sphuta-siddhānta* of **Brahmagupta,** and the name *Sindhind* is derived from the word *Siddhānta.* A Persian named **Yaqub ibn Tariq** also went to the court of the Caliphs about this time and probably assisted in translating the works of Brahmagupta.

and astronomy came to be known to the scholars at Bagdad. This was known as *Sindhind* and contained the important Hindu table of sines. After this time to the year 1000 A.D. very little progress was made in India. **Mahavira** (fl. 850 A.D.) seems to have made efforts to improve upon the works of Brahmagupta. In the meantime the knowledge of India passed into the keeping of Arabs. The chief Arab writer on astronomy was **Albategnius** (fl. 920 A.D.). Like the Hindus he used half chords instead of chords. Some mathematicians are of opinion that he discovered the cosine formula, but there is no evidence to show that he had any real knowledge of spherical trigonometry. In fact, he borrowed it from the Hindu astronomy. **Abu'l Wefa** (940—998 A.D.) and his contemporary **Abu Nasr** tried to systematise the older knowledge but it was the Persian astronomer, **Nasir ed-din al-Tusi** (1201—1274 A.D.), whose work *Shakl al-qattâ* reveals trigonomtry as a science by itself. Among the Hindu writers from 1000—1500 A.D., the first was **Sridhara** who was born in 991 A.D. but he did not contribute much to the science of Spherical Trigonometry. The other writer of prominence is **Bhaskara** (1114—1185 A.D.), whose *Siddhānta Siromani* contains a book, *Goladhia*, devoted to astronomy and sphericity of the earth. He gave a method of constructing a table of sines for every degree. With the decline of Bagdad, the study of spherical triangles, for astronomical work, assumed greater importance in

Spain. **Gabir ben Aflah** of Sevilla (1140 A.D.) wrote on spherical trigonometry and introduced the "rule of four quantities." By the 14th century England came to know of the Hindu Trigonometry through the Arab Trigonometry.

Among the modern writers to exhibit Trigonometry as a science, independent of Astronomy, was the German mathematician Johann Müller, better known as **Regiomontanus** (1436—1476). His work *De triangulis omnimodis Libri V*, written in 1464, may be said to have laid the foundation for later works on plane and spherical trigonometry. **Copernicus** (1473—1543) completed some of the works left unfinished by Regiomontanus, in his *De Lateribus et Angulis Triangulorum* (1542). The Danish astronomer **Tycho Brahe** also gave the cosine formula in 1590. With the French mathematician **Vieta** (1540—1603) began the first systematic development of the calculation of plane and spherical triangles. The theorem for cosine of angles was given by Vieta in 1593. The cotangent theorem was given in substance by him but was afterwards proved by Snellius in 1627. The name Trigonometry first appeared in an important work on Trigonometry by the German mathematician **Pitiscus** (1561—1613) in 1595. **Albert Girard** (1595—1632) published at the Hague, in 1626, a noteworthy work on Trigonometry, in which he made use of the spherical excess, in finding the area of a spherical triangle. This also appeared in his *Invention nouvelle en l'Algébre* in 1629. The area of a spherical triangle was also given by **Cavalieri**

(1598—1647) in his *Directorium generale* (Bologna, 1632), and afterwards in his *Trigonometria plana et spherica* (Bologna, 1643). **Napier** (1550—1617) replaced the rules for spherical triangles by one clearly stated rule, the Napier's analogies, published in his *Mirifici Logarithmorum canonis Descriptio* in 1614. He also gave two rules of circular parts, which included in them all the formulae for right-angled spherical triangles. The properties of the polar triangles were discovered by **Snellius** (1591—1626 A.D.) in his *Trigonometria*, published posthumously at Leyden in 1627. **Euler** (1707—1783 A.D.) gave a fresh impetus to the study of the subject by publishing several memoirs in the Royal Academy of Berlin and in the *Acta Petropolitana*. **Delambre** published his analogies in 1809. Valuable contributions to the subject were also made by **Lagrange** (1736—1813), **Lhuilier** (1750—1840), **Legendre** (1752—1833), **Gauss** (1777—1855), **Lexell** (1782), **Chasles** (1831), **Schulz** (1833), **Gudermann** (1835), **Borgnet** (1847), **Neuberg**, **Von Staudt** (1798-1867) and **Simon Newcomb** (1835—1909), **E. Study** (1893) and **F. Meyer.**

The case for spherical triangles with sides and angles not necessarily less than π is generally ascribed to **Mobius** * but it seems that Gauss † had not only thought of this generalisation, but had worked it out.

* See *Gesellschaft der Wissenschaften zu Leipzig*, 1860, p. 51.

† See, *Theoria motus Corporum Coelestium*, 1809, § 54.

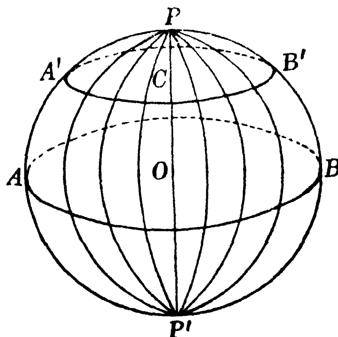
SPHERICAL TRIGONOMETRY

CHAPTER I

SPHERE

1.1. Sphere. A sphere is a solid figure such that every point of its surface is equally distant from a fixed point within it, which is called the *Centre of the Sphere.*

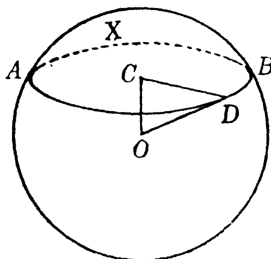
Any straight line joining the centre of a sphere to any point on its surface is called a *Radius*, and the straight line drawn through the centre and terminated both ways by the sphere is called a *Diameter of the Sphere.*



A sphere can be generated by the revolution of a circle round any of its diameter as axis.

1.2. Intersection of Sphere by a plane. If a plane intersects a sphere, the resulting section will be some curve on the surface of the sphere and we prove below that

The section of the surface of a sphere by a plane is a circle.



Let ABX be the section of the sphere made by a plane and let O be the centre of the sphere. Draw OC perpendicular to the plane of ABX. Take any point D on the section ABX and join CD and OD. Now OCD is a right-angled triangle, for OC is perpendicular to the plane ABX and hence perpendicular to CD. Therefore $CD^2 = OD^2 - OC^2$. But OD is constant being radius of the sphere and OC is constant for O and C are fixed points, and hence CD is of constant length. Thus any point D in the section ABX is equally distant from the fixed point C in its plane, that is, ABX is a circle of which C is the centre.

1.3. Great Circle and Small Circle.* When the plane intersecting the sphere passes through the centre of the sphere, its circular section is called a *Great Circle*, thus AB is a *Great Circle*. When the section does not pass through the centre it is called a *Small Circle*, thus $A' B'$ is a *Small Circle*. (See figure of Art. 1.1.)

The solid cut off by the plane of a great circle is called a *Hemisphere*, and that cut off by the plane of a small circle is called a *Segment of the sphere.*

Note 1.—Only one great circle can be drawn through two given points on the surface of a sphere, for its plane must pass through the centre of the sphere, and three non-collinear points uniquely determine a plane. The great circle is unequally divided at the two points, and by the arc joining the two points we shall always mean the smaller of the two. But if the two given points be the extremeties of a diameter, an infinite number of great circles can be drawn through them. (See figure of Art. 1.1.)

Note 2.—The shortest arc that can be drawn on the surface of a sphere joining two points on it, is the great circular arc through them, for the shortest arc must have the least curvature, and so it must belong to the circle of the greatest radius, *i.e.*, the great circle.

1.4. Axis and Poles. The *Axis* of a circle on a sphere is that diameter of the sphere which is perpendicular to the plane of the circle. The extremeties

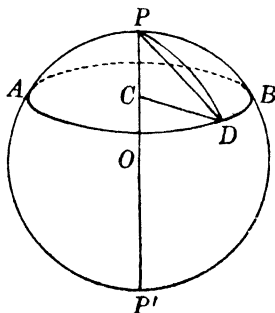
* The nomenclature is perhaps due to the fact that the radius of a great circle (which is the same as the radius of the sphere) is always greater than that of any small circle, as is evident from the relation $CD^2 = OD^2 - OC^2$ of Art. 1.2.

of the axis are called the *Poles** of the circle. Thus if PP' (fig. of Art. 1.5) is perpendicular to the small circle AB, P and P' are its poles, of which the nearer pole P will usually be denoted as *the pole.* The poles of the great circle are equidistant from the plane of the great circle. Any point and the great circle of which it is the pole are termed *pole and polar* with respect to each other.

EXAMPLE

Shew that the line joining the centre of the sphere to the pole of a small circle passes through its centre.

1.5. Theorem. *The pole of a circle is equidistant from every point on the circumference of the circle.*



Let O be the centre of the sphere and AB any circle on it of which C is the centre, and P and P' are the poles. Take any point D on AB. Join CD and PD. Then $PD^2 = PC^2 + CD^2 =$ constant.

* The expression *pole of a circle* is due to **Archimedes** of Syracuse (287-212 B.C.).

Now as the chord PD is constant, therefore the arc of the great circle intercepting PD is also constant for all positions of D on the circle AB. Thus the distance of the pole of a circle from every point on its circumference is constant whether the distance be measured by a straight line or by a great circular arc.

The great circular arc PD joining the pole P of the circle AB to any point D on its circumference, is called the *Spherical Radius* of the circle AB. The spherical radius of a great circle is a quadrant. (See Art. 1.9.)

1.6. Theorem. *Two great circles bisect each other.*

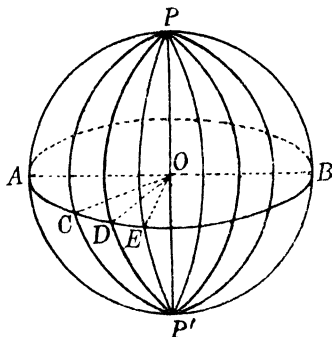
The plane of each great circle passes through the centre of the sphere. Hence the line of intersection of these planes is a diameter of sphere, as also of each great circle. Therefore the great circles are bisected at the points where they meet.

1.7. Angle between two circles. When two circles intersect, the angle between the tangents at either of their points of intersection is called the angle between the circles. If these circles are great circles, their planes pass through the centre, and their line of intersection is a diameter of the sphere to which the tangents are perpendicular and hence the angle between the tangents is the angle of intersection of the planes.* Thus

* When two planes intersect, the angle between them is measured by the angle between any two straight lines drawn one in each plane, at any point on their line of intersection and perpendicular to it.

The angle of intersection of two great circles is equal to the inclination of their planes.

1.8. Secondary Circles. Great circles which pass through the poles of another great circle are called *Secondaries* to that circle, which again is termed *Primary* circle in relation to them. Thus, in the figure, ABC is the primary circle and all the circles through P and P' are secondaries to it. It is evident that there can be an infinite number of such secondaries, the planes of which intersect in the line PP', the axis of the primary circle.



Since PP' is perpendicular to the plane ABC, any plane passing through PP' is also perpendicular to the plane ABC. Hence

Any great circle and its secondary cut each other at right angles.

Again since PO is perpendicular to OA and OC, AOC is the angle of inclination of the planes of PA and PC, and this is measured by the arc AC. Hence the angle between the circles PA and PC is measured by the arc AC, *i.e.*,

The angle between any two great circles is measured by the arc intercepted by them on the great circle to which they are secondaries.

1.9. Theorem. *The arc of a great circle which is drawn from a pole of a great circle to any point in its circumference is a quadrant.* (Fig. of Art. 1.8.)

Let P be a pole of the great circle ABC and O the centre of the sphere. Join PO. Then PO is perpendicular to the plane ABC and hence perpendicular to OA, OB, OC and OD. Hence each of the angles POA, POB, POC and POD is a right angle, *i.e.*, the arc PA, PB, PC or PD is a quadrant.

1.10. The Converse Theorem. *If the arcs of great circles joining a point on the surface of a sphere with two other points on it, which are not opposite extremities of a diameter, be each a quadrant, then the first point is a pole of the great circle passing through the other two.*

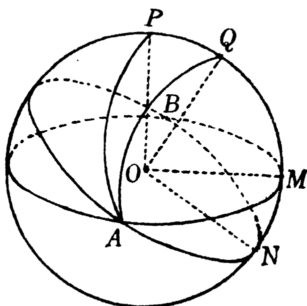
For if PA and PC (fig. of Art. 1.8) be each a quadrant, the angles POA and POC are right angles. Therefore PO is perpendicular to OA and OC, and

hence perpendicular to the plane AOC, *i.e.*, P is a pole of the great circle AC.

1.11. Theorem. *If two arcs of great circles, which are not parts of the same great circle, be drawn from a point on the surface of a sphere such that their planes are at right angles to the plane of a given circle, then that point is a pole of the given circle.*

Since the planes of the two arcs are at right angles to the plane of the given circle, their line of intersection is also perpendicular to the plane of the given circle; and as it passes through the centre of the sphere, it is the axis of the given circle. Hence the given point is a pole of the circle.

1.12. Theorem. *The points of intersection of two great circles are the poles of the great circle passing through the poles of the given circles.*



Let the two great circles intersect at A and B, and let P and Q be their poles. Join PA and QA.

Then PA and QA are each a quadrant (Art. 1.9) and hence A is the pole of the great circle PQ (Art. 1.10). Similarly B is the other pole.

✓**1.13. Theorem.** *The angle between two great circles is equal to the angular distance between their poles.*

For, taking the figure of the last article, A is the pole of the circle PQ; hence AM and AN are each a quadrant. The angle between the circles AMB and ANB is measured by the arc MN (Art. 1.8). Also PM and QN are quadrants and the angular distance of the poles is measured by the arc PQ. Therefore

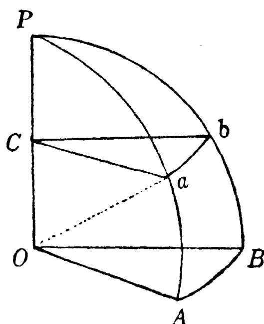
$$\text{arc } PQ = PM - QM = QN - QM = MN.$$

Since these arcs are equal, they will subtend equal angles at the centre. Hence joining OP, OQ, OM and ON, we have angle POQ = angle MON, *i.e.*, the angle subtended at the centre of the sphere by the arc of a great circle joining the poles of two great circles is equal to the inclination of their planes.*

1.14. *To compare the arc of a small circle subtending any angle at its centre with the arc of a great circle subtending an equal angle at the centre of the sphere.*

* It is obvious that the angle between the two planes is equal to the angle between their perpendiculars OP and OQ.

Let ab be the arc of a small circle whose centre is C and whose pole is P. Let O be the centre of the sphere. Then OP is at right angles to the plane aCb. OP is also at right angles to the plane of the great circle of which P is a pole. Through P draw great circles PaA and PbB to meet this great circle at A and B. Then OP is perpendicular to OA, OB, Ca and Cb. Hence either of the angles aCb or AOB measures the angle between the planes POA and POB and therefore $\angle aCb = \angle AOB$.



Hence $$\frac{\text{arc } ab}{\text{radius } Ca} = \frac{\text{arc } AB}{\text{radius } OA}$$

or $$\frac{\text{arc } ab}{\text{arc } AB} = \frac{Ca}{OA} = \frac{Ca}{Oa} = \sin P\hat{O}a = \cos A\hat{O}a.$$

Thus $$\text{arc } ab = \text{arc } AB \cos A\hat{O}a.$$

i.e., Distance between two places on the same parallel of latitude = Difference in their longitude multiplied by cosine of their common latitude.

EXAMPLE WORKED OUT

On a sphere whose radius is r a small circle of spherical radius, θ, is described, and a great circle is described having its pole on the small circle, show that the length of their common chord is

$$\frac{2r}{\sin\theta}\sqrt{-\cos 2\theta}$$

(*Science and Art Exam. Papers.*)

Let O be the centre of the sphere and C the centre of the small circle. Then OC is perpendicular to the plane of the small circle. Take any point P on the small circle as the pole of the great circle. Then

$$\angle POC = \text{the angular radius of the small circle} = \theta$$

and hence $OC = r\cos\theta$ and $CP = r\sin\theta$, when r is the radius of the sphere.

Let c be the length of the common chord and d the length of the perpendicular from C on it. Then

$$\left(\frac{c}{2}\right)^2 = r^2\sin^2\theta - d^2$$

Again since the angle between OC and the plane of the great circle is $90^\circ - \theta$, we have

$$\cot\theta = \frac{d}{r\cos\theta} \quad \text{or} \quad d = r\cos\theta\cot\theta.$$

Therefore $\left(\frac{c}{2}\right)^2 = r^2\sin^2\theta - r^2\cos^2\theta\cot^2\theta$

$$= \frac{r^2(\sin^2\theta - \cos^2\theta)}{\sin^2\theta}$$

Or, $c = \dfrac{2r}{\sin\theta}\sqrt{-\cos 2\theta}$

N.B.—For a real section 2θ must be greater than 90° and hence the negative sign under the radical sign.

EXAMPLES

1. Shew that any great circle is the locus of the poles of all its secondaries.

2. Shew that the angle between the plane of any circle and the plane of a great circle which passes through its poles is a right angle.

3. Two equal small circles are drawn touching each other. Shew that the angle between their planes is twice the complement of their spherical radius.

(*Science and Art Exam. Papers.*)

4. The angle subtended at the centre of a circle by two points on it is equal to the angle subtended by them at its pole.

5. If two great circles are equally inclined to a third, their poles are equidistant from the pole of the third.

6. If a point is equidistant from three great circles, it is also equidistant from their poles.

7. If two spheres intersect each other, shew that their curve of intersection is a circle.

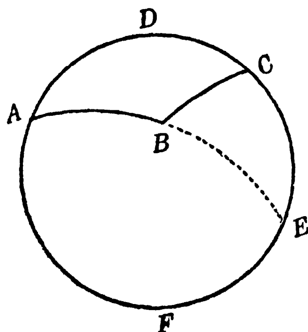
CHAPTER II

Spherical Triangle

2.1. Spherical triangle. A spherical triangle is a triangle formed by three arcs of great circles on the surface of a sphere. The arcs are spoken of as the sides, and their angles of inclination at the points where they meet, the angles of the spherical triangle. As in plane trigonometry, the angles are usually denoted by the letters A, B, C and their opposite sides by the letters a, b and c. The angles and the sides are sometimes spoken of as *elements* or *parts* of a spherical triangle. Unless stated to the contrary, all arcs drawn on the surface of a sphere will be taken to be arcs of great circles.

2.2. Restriction of the sides and the angles. Two points on the surface of a sphere may be taken to be joined by either of the two segments of the great circle passing through them. Hence we can have eight triangles having for their vertices A, B and C. So to avoid ambiguity and to simplify our study it has been conventional (as in Art. 1.3, note 1) to mean by any of its sides, the lesser segment of the great circle passing through the two corresponding vertices. Thus we get one triangle ABC each *side* of which *is less than a semicircle*, and we denote this particular triangle as the spherical

triangle ABC. Thus in the figure, triangle ABC is that one formed by the arcs ADC, AB and BC.



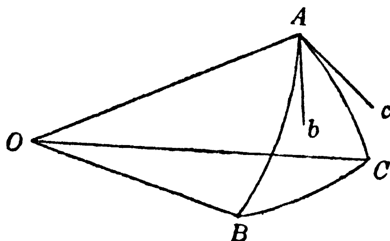
It follows from the above that *each angle of a spherical triangle must be less than two right angles.*

For consider the triangle ABC having the angle B greater than two right angles. Produce the arc AB to meet the circle ACF at E. Then the arc AFE is a semicircle and hence the arc AEC is greater than a semicircle. Thus the triangle ABC having the angle B greater than two right angles is formed by the arcs AB, BC and AEC of which the latter is greater than two right angles. Such a triangle we have excluded from our consideration. Hence we conclude that

The sides and the angles of a spherical triangle must each be less than two right angles.

The sides and the angles of a spherical triangle will generally be expressed in circular measure.

2.3. Formation of a Spherical triangle. Let O be the centre of the sphere and suppose three planes form a solid angle at O. These planes intersect the surface of the sphere in arcs of great circles AB, BC and CA which form the sides of the spherical triangle ABC.



Now the plane angle $AOB = \dfrac{\text{arc } AB}{\text{radius } OA}$,

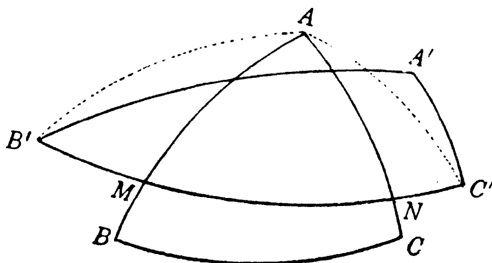
angle $BOC = \dfrac{\text{arc } BC}{\text{radius } OB}$ and angle $AOC = \dfrac{\text{arc } AC}{\text{radius } OC}$,

and as $OA = OB = OC$, we see that the arcs AB, BC and CA are proportional to the plane angles AOB, BOC and COA, which they subtend at the centre of the sphere.

If Ab and Ac are tangents to the arcs AB and AC respectively, the angle A is equal to the angle bAc, which again is the angle between the planes AOB and AOC containing the sides AB and AC. Thus *the angles of a spherical triangle are the same as the inclination of the plane faces forming the solid angle at the centre O of the sphere.*

2.4. Polar Triangle.* If a triangle is formed with the poles of the sides of a given triangle as its angular points, it is called a *Polar Triangle* with respect to the given triangle. Thus if ABC be a given spherical triangle and A', B', C' be the poles of BC, CA and AB, then $A'B'C'$ is the polar triangle of ABC. The triangle ABC is called the **primitive triangle** with respect to $A'B'C'$. Since there are two poles for each side, we should get eight such polar triangles with respect to the given triangle. But we call that particular one to be the polar traingle in which the poles A', B', C' lie on the same side of their polars as the opposite angles A, B, C.

2.5. Theorem. *If one triangle be the polar triangle of another, then the latter will be the polar triangle of the former.*



Let ABC be a given triangle and $A'B'C'$ be the polar triangle. Join AB' and AC'.

* The properties of the polar triangle were discovered by Snellius (1591-1626 A.D.). His *Trigonometria* was published (posthumously) at Leyden in 1627.

Now since B' is the pole of AC, the arc AB' is a quadrant, and since C' is the pole of AB, the arc AC' is also a quadrant. Hence A is a pole of $B'C'$. And since A and A' lie on the same side of BC, AA' is less than a quadrant. Again as A is a pole of $B'C'$, and AA' is less than a quadrant, A and A' lie on the same side of $B'C'$. Similarly B is the pole of $A'C'$ and C is the pole of $A'B'$, and B, B' lie on the same side of $A'C'$, and C, C' on the same side of $A'B'$. Therefore ABC is the polar triangle of $A'B'C'$.

2.6. Theorem. *The sides and angles of the polar triangle are respectively the supplements of the angles and sides of the primitive triangle.*

Let M and N be the points of intersection of AB and AC by $B'C'$ (see fig. of Art. 2.5). Then AM and AN are each a quadrant, because A is the pole of $B'C'$; and the angle A is measured by the arc MN. Again $B'N$ and $C'M$ are also quadrants. Hence

$$B'N + C'M = B'C' + MN = 2 \text{ right angles},$$

or $$B'C' = \pi - A.$$

Similarly $$A'C' = \pi - B \quad \text{and} \quad A'B' = \pi - C.$$

Again since ABC is the polar triangle of $A'B'C'$, we have

$$BC = \pi - A', \quad CA = \pi - B', \quad \text{and} \quad AB = \pi - C'.$$

Hence denoting the sides of the triangle $A'B'C'$ by a', b', c', we have

$$a' = \pi - A, \quad b' = \pi - B, \text{ and } c' = \pi - C.$$

and $$A' = \pi - a, \quad B' = \pi - b, \text{ and } C' = \pi - c.$$

Note.—From the above property polar triangles are also termed *Supplemental triangles.* Any theorem involving the sides and angles of a spherical triangle necessarily holds good for the polar triangle also. Hence for any such theorem there is a supplemental theorem involving the opposite angles and sides, and it is obtained by changing the sides and angles of the original theorem into the supplements of the corresponding angles and sides respectively.

2.7. Theorem. *Any two sides of a spherical triangle are together greater than the third side.*

Let ABC be a spherical triangle and O the centre of the sphere. Now any two of the three plane angles forming the solid angle at O is greater than the third. Thus

$$\angle AOB + \angle BOC > \angle AOC$$

or, $$\frac{AB}{OA} + \frac{BC}{OA} > \frac{AC}{OA},$$

that is, the sum of the arcs AB and BC is greater than the arc AC.

Cor. Any one side of a spherical polygon is less than the sum of all the others.

Example

Shew that the difference of any two sides of a spherical triangle is less than the third side.

2.8. Theorem. *The sum of the three sides of a spherical triangle is less than the circumference of a great circle.*

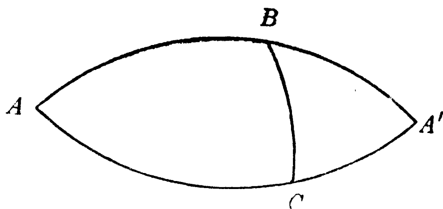
Let ABC be a spherical triangle, and O the centre of the sphere. The sum of the plane angles AOB, BOC and COA forming the solid angle at O is less than 2π.

i.e., $$\frac{AB}{OA}+\frac{BC}{OA}+\frac{CA}{OA}<2\pi$$

or $$AB + BC + CA < 2\pi . OA.$$

Thus the sum of the sides is less than the circumference of a great circle. The angular measure of the sum of the sides is less than four right angles.

Aliter.



Let the sides AB and AC be produced to meet at the point A'. Then the arcs ABA' and ACA' are semicircles. Now any two sides of the triangle $A'BC$ are together greater than the third. Hence we have

$$A'B + A'C > BC.$$

Therefore, $$AB+A'B+AC+A'C > AB+BC+AC.$$

or, $$ABA'+ACA' > AB+BC+CA$$

i.e., the sum of the sides is less than the circumference of a great circle.

Note.—The above proposition can be easily extended in the case of polygons.

2.9. Theorem. *The sum of the three angles of a spherical triangle is greater than two right angles and less than six right angles.*

Let ABC be a spherical triangle. Since each of the angles A, B and C is less than π, we have

$$A+B+C < 3\pi.$$

Again $a'+b'+c' < 2\pi$, where a', b' and c' are the sides of the polar triangle of ABC. But $a'=\pi-A$, $b'=\pi-B$, $c'=\pi-C$ (Art. 2.6).

Hence, $\pi-A+\pi-B+\pi-C < 2\pi$.

or, $A+B+C > \pi$

Thus $\pi < A+B+C < 3\pi$.

2.10. Theorem. *The difference between any two angles of a spherical triangle is less than the supplement of the third angle.*

Let ABC be a spherical triangle and $A'B'C'$ be its polar triangle. Now any two sides of $A'B'C'$ are together greater than the third.

Hence, $a' + b' > c'$

or, $\pi - A + \pi - B > \pi - C$

i.e., $A + B < \pi + C$

Hence, $A - C < \pi - B$

and $B - C < \pi - A.$

Similarly, $A - B < \pi - C.$

This theorem gives the limit of the third angle when two angles are given.

EXAMPLES

1. Given two angles of a spherical triangle to be 145° and 80°, find the limit of the third angle.

Here $A = 145°$ and $B = 80°$

Hence $145° - 80° = 65° < \pi - C$

or $C < 180° - 65°$, *i.e.*, less than 115°.

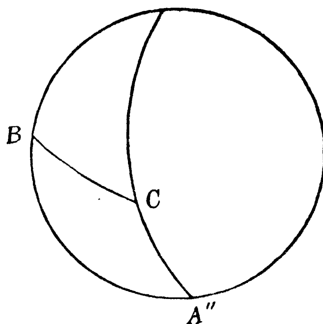
2. If the difference between any two angles be 90°, shew that the remaining angle is less than 90°.

3. Shew that the difference of the oblique angles of a right-angled triangle is less than a right angle.

4. Shew that the sum of the angles of a right-angled triangle is less than four right angles.

2.11. Lune. A *Lune* is a portion of the surface of a sphere enclosed by two great semicircles. Thus

in the figure, the semicircles ABA'' and ACA'' enclose a lune. A'' is the point diametrically opposite to A.



The angle BAC is called the *Angle of the Lune.* The triangles ABC and $A''BC$ are called *Colunar Triangles*, because they together make up a lune.

The area of a lune can be easily expressed in terms of its angle, for,

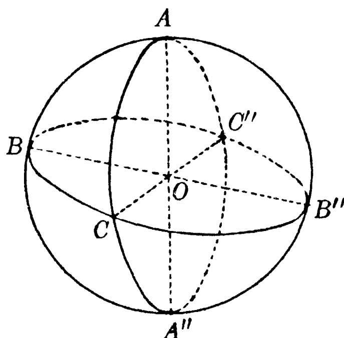
$$\frac{\text{Area of Lune}}{\text{Area of Sphere}} = \frac{\text{Angle of Lune}}{2\pi}$$

or, $$\text{Area of Lune} = 4\pi r^2 \frac{A}{2\pi} = 2Ar^2,$$

where r is the radius of the sphere and A the circular measure of the angle BAC.

If B'' and C'' be points diametrically opposite to B and C respectively, we get two other colunar triangles of ABC, namely $B''CA$ and $C''AB$.

2.12. Antipodal triangles. If the points A'', B'' and C'' diametrically opposite to A; B and C respectively be taken to form a triangle, the triangle $A''B''C''$ is called the *Antipodal triangle* to ABC.



The arcs AB and $A''B''$ join diametrically opposite points. Hence they are parts of the same great circle and are equal in length. So also the arcs AC and $A''C''$ are equal, as also the arcs BC and $B''C''$. Again the angle A is equal to the angle A'' for they are comprised by the great circles $ABA''B''$ and $ACA''C''$. Similarly $B=B''$ and $C=C''$. Hence the triangles ABC and $A''B''C''$ have all their elements equal. If the triangle $A''B''C''$ be shifted from its place on the surface of the sphere till B'' falls on B and C'' falls on C, the point A'' will not fall on A but will lie on the opposite side of BC. That is the triangle $A''B''C''$ is not superposable on triangle ABC. Such triangles are

called *symmetrically equal* * as distinguished from *identically equal or congruent* triangles which are superposable on each other.

2.13. Two triangles on the same sphere are equal (symmetrically or identically) when they have the following elements of one triangle equal to the corresponding elements of the other triangle.

(1) Two sides and the included angle,
or, (2) Three sides,
or, (3) Two angles and the adjacent side.
or, (4) Three angles.

The cases (1) to (3) are analogous to plane geometry, but (4) has no such analogue. It is derived from (2) by the consideration of the supplemental triangles.

2.14. In this and the following articles are given some theorems of plane geometry which hold good in the case of spherical triangles as well. One such case has already been dealt with in Art. 2.7.

Theorem. *The angles at the base of an isosceles spherical triangle are equal, and conversely if two angles of a spherical triangle are equal, the opposite sides are equal.*

* This term is due to **Legendre** (1752-1833). See his *E'léments de Géométrie*, Paris, VI, Def. 16, 1794.

Let ABC be a spherical triangle of which the sides AB and AC are equal. Take D to be the middle point of AC. Join AD by a great circular arc. Then the triangles ADB and ADC have their corresponding sides equal, each to each, and therefore they are symmetrically equal, Hence the angle $B =$ the angle C.

For the converse case, take the angle $B =$ the angle C, and let $A'B'C'$ be the polar triangle of ABC.

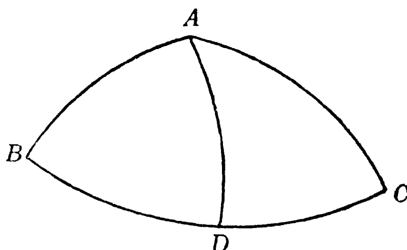
Now $b' = \pi - B$ and $c' = \pi - C$

and as $B = C$, we have $b' = c'$; hence $B' = C'$.

Again $b = \pi - B'$ and $c = \pi - C'$.

Therefore $b = c$, *i.e.*, AB and AC are equal.

2.15. Theorem. *If one angle of a spherical triangle is greater than another, then the side opposite to the greater angle is greater than the side opposite to the less and conversely.*



Let ABC be a triangle of which the angle A is greater than the angle B. Draw a great circular arc

AD making the angle BAD = the angle ABD. Then the arc AD = the arc BD.

But in the triangle ADC, $AD+DC > AC$.
Therefore $BD+DC$, *i.e.*, $BC > AC$,

The converse case is easily proved with the help of the polar triangles.

EXAMPLES

1. When does a polar triangle coincide with the primitive triangle ?

Ans. When each element equals $\frac{1}{2}\pi$.

2. If two small circles on a sphere touch each other, shew that the great circle joining their poles passes through their point of contact.

3. If a triangle is equilateral, shew that its polar triangle is also equilateral.

4. If two sides of a spherical triangle be quadrants, shew that the angles at the base are right angles.

5. If all the sides of a spherical triangle be quadrants, all of its angles are right angles.

6. If two sides of a triangle are supplemental, shew that the opposite angles are also supplemental.

7. If two sides of a triangle are supplemental, shew that two sides of its polar triangle are also supplemental.

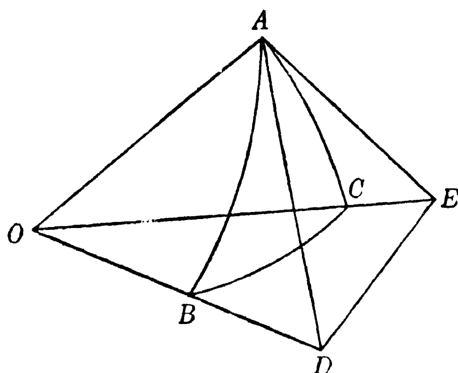
8. The base of a spherical triangle is given : find the locus of the vertex when the sum of the other two sides is equal to two right angles.

Ans. A great circle having the middle point of the base as pole.

CHAPTER III

RELATIONS BETWEEN THE TRIGONOMETRICAL FUNCTIONS OF THE SIDES AND THE ANGLES OF A SPHERICAL TRIANGLE.

✓3.1. **Fundamental Formulae.** *Expression for the cosine of an angle in terms of the sines and cosines of the sides.*



Let ABC be a spherical triangle and O the centre of the sphere. At A draw the tangents AD and AE to the arcs AB and AC respectively. They lie in the planes AOB and AOC respectively. Let them meet OB and OC produced at the points D and E. Then the angle EAD is equal to the angle A of the spherical triangle. Join DE.

From the triangle DOE, we have

$$DE^2 = OD^2 + OE^2 - 2\ OD.OE \cos a.$$

Again from the triangle DAE, we have

$$DE^2 = AD^2 + AE^2 - 2\ AD.AE \cos A.$$

Hence by subtraction we have

$$0 = OD^2 - AD^2 + OE^2 - AE^2 + 2\ AD.AE \cos A - 2\ OD.OE \cos a.$$

$$= 2\ OA^2 + 2\ AD.AE \cos A - 2\ OD.OE \cos a$$

for the angles OAD and OAE are right angles.

Therefore, $\cos a = \dfrac{OA}{OE}.\dfrac{OA}{OD} + \dfrac{AE}{OE}.\dfrac{AD}{OD} \cos A,$

i.e., $\cos a = \cos b \cos c + \sin b \sin c \cos A.$

Similarly, $\cos b = \cos c \cos a + \sin c \sin a \cos B,$

and $\cos c = \cos a \cos b + \sin a \sin b \cos C.$

Hence $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c},$

$$\cos B = \frac{\cos b - \cos c \cos a}{\sin c \sin a},$$

and $\cos C = \dfrac{\cos c - \cos a \cos b}{\sin a \sin b},$

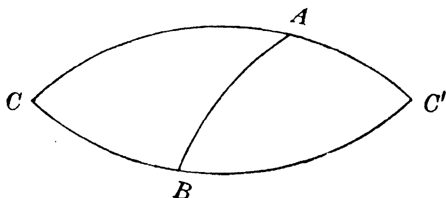
The above relations are the fundamental formulæ *

* The cosine theorem was implied in the rules of ancient Hindu Mathematicians for finding the time-altitude and the alt-azimuth equations and the diurnal motion, and was used by them to solve spherical triangles. *Cf. Pañca Siddhāntikā*, IV, 42-44, by **Varahamihira** (505-587); *Brāhma Sphūta Siddhānta*, III, 26-40; and *Khandakhādyaka*, III, 13, by **Brahma Gupta** born in 598 A.D.; and *Sūryasiddhānta*, III, 34-35 (written about the 4th century). It was exhibited in a systematic form by

of the spherical trigonometry. All other formulæ can be made to depend upon them.*

3.2. On referring to the figure of the last article it is seen that the angles AOD and AOE are acute angles, and hence the arcs b and c containing the angle A are each less than a quadrant. No such restriction, however, has been placed upon the arc a, so that a may be greater than, equal to, or less than a quadrant. We shall now show that the above formulæ apply to all spherical triangles whether the arcs be greater than, equal to or less than a quadrant.

(1) Let one side b be greater than a quadrant.



Produce CA and CB to meet at C'. Then

the German Mathematician **Regiomontanus** (1436-1476) in 1460 and afterwards by the Danish Astronomer **Tycho Brahe** about 1590. **Euler** also gave a proof of the theorem in his *Mémoires de Berlin* in 1753. Some are of opinion that it was discovered by **Albategnius** (900 A.D.) who in fact borrowed it from the Hindu Astronomy.

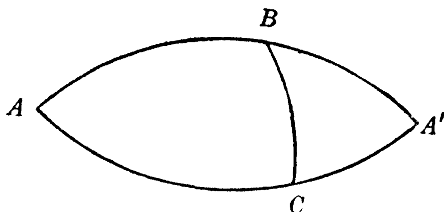
* As was shown by **Lagrange** (1736-1813). See also **Gauss** (1777-1855), *Ges. Werke*, Vol. IV, p. 401.

$C'A=\pi-b$, and $C'B=\pi-a$. Hence from the triangle ABC', we have

$$\cos BC'=\cos AB \cos AC'+\sin AB \sin AC' \cos BAC',$$

or $\quad \cos a=\cos b \cos c+\sin b \sin c \cos A.$

(2) Next let b and c be each greater than a quadrant.



Produce AB and AC to meet at A'. Then from the triangle $A'BC$, we have

$$\cos BC=\cos A'C \cos A'B+\sin A'C \sin A'B \cos A',$$

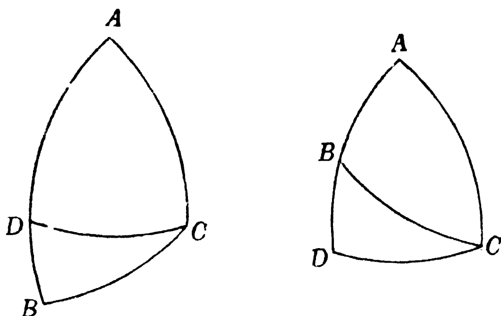
or $\quad \cos a=\cos b \cos c+\sin b \sin c \cos A,$

for $A'C=\pi-b$, $A'B=\pi-c$ and $A=A'$.

(3) Thirdly let b be equal to a quadrant.

From AB or AB produced cut off AD equal to a quadrant. Join CD.

Now if CD be a quadrant, C will be the pole of



AB, and the formula becomes $0=0$. If CD be not a quadrant, we have from the triangle BCD,

$$\cos a = \cos BD \cos CD + \sin BD \sin CD \cos BDC$$

$$= \sin c \cos A$$

for $\cos BDC = 0$. The formula also reduces to this when $b = \frac{1}{2}\pi$.

(4) Lastly let $b = c = \frac{1}{2}\pi$. Then our formula reduces to $\cos a = \cos A$, as is otherwise evident, since A is the pole of BC. Thus $A = a$.

Thus our formula is universally true.

3.3. Expression for the sine of an angle.

We have $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$.

Therefore

$$\sin^2 A = 1 - \left\{\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right\}^2$$

$$= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{(1-\cos^2 b)(1-\cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 b \sin^2 c}$$

so that

$$\sin A = \frac{\sqrt{\{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c\}}}{\sin b \sin c}.$$

As $\sin A$, $\sin b$ and $\sin c$ are all positive, the radical must be taken with the positive sign.

For the sake of brevity and owing to the importance of the expression under the radical sign, we put

$$4n^2 = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c,$$

so that

$$\sin A = \frac{2n}{\sin b \sin c}, \quad \sin B = \frac{2n}{\sin c \sin a},$$

and

$$\sin C = \frac{2n}{\sin a \sin b}.$$

n is called the *norm of the sides of the spherical triangle.**

* This nomenclature is due to Professor **Neuberg** of Liege. Professor Von **Staudt** (1798-1867) calls $2n$ *the sine of the triangle* ABC. See *Crelle's Journal*, XXIV, 1842, p. 252.

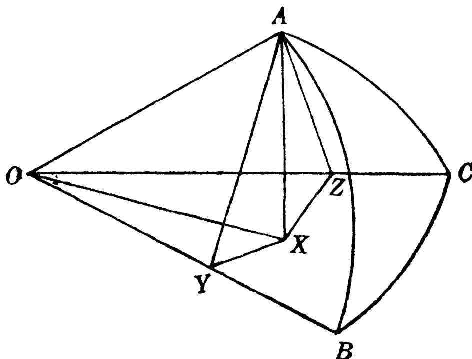
3.4. From the value of sin A, we have at once

$$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}=\frac{2n}{\sin a \sin b \sin c},$$

i.e., the sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.

Owing to the importance of this result, we give an independent proof of it in the next article.

3.5. Rule of Sines. *The sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.*



Let ABC be a spherical triangle and O the centre of the sphere. From A draw AX perpendicular to the plane BOC, and AY and AZ perpendiculars on OB and OC respectively. Join OX, XY and XZ.

Then since AX is perpendicular to the plane BOC, it is at right angles to OX, XY and XZ.

Hence $OA^2 = OX^2 + AX^2,\ AY^2 = AX^2 + XY^2$

and $AZ^2 = AX^2 + XZ^2$.

Also $OA^2 = OY^2 + AY^2 = OZ^2 + AZ^2$.

Therefore, $OX^2 = OA^2 - AX^2 = OY^2 + AY^2 - AX^2$

$= OY^2 + XY^2$.

Similarly, $OX^2 = OA^2 - AX^2 = OZ^2 + AZ^2 - AX^2$

$= OZ^2 + XZ^2$.

Thus XY and XZ are at right angles to OB and OC respectively.

Now since AY and XY are in the planes OAB and OBC and are at right angles to their line of intersection OB at Y, the angle AYX measures the angle B of the spherical triangle. (Art. 2.3). Similarly angle AZX measures the angle C. Hence

$$AX = AY \sin AYX = AY \sin B = OA \sin c \sin B$$

and $$AX = AZ \sin AZX = AZ \sin C = OA \sin b \sin C.$$

Therefore $$\frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Since B and C are any two angles of the spherical triangle, it follows that

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \text{ *}$$

* This theorem appears in a different form in the 3rd book of the *Sphaerica* of **Menelaus** of Alexandria (100 A.D.). It was also known to **Abû'l Wefâ** (940-998) of Arabia and possible to his contemporary **Abû Nâsr.**

3.6. Analogous formulæ in Plane Trigonometry.

The sine and cosine formulæ in the previous articles bear some resemblance to the corresponding formulæ in Plane Trigonometry. In fact the latter can be derived from the former when r the radius of the sphere is taken to be indefinitely great, for then the great circular arc reduces to a straight line and the limiting form of the proposed formula becomes the formula for Plane Trigonometry.

Let α, β, γ be the lengths of the sides of the spherical triangle ABC, then $\frac{\alpha}{r}$, $\frac{\beta}{r}$, $\frac{\gamma}{r}$ are the circular measures of the sides, r being the radius of the sphere. From Art. 3.1 we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos \frac{\alpha}{r} - \cos \frac{\beta}{r} \cos \frac{\gamma}{r}}{\sin \frac{\beta}{r} \sin \frac{\gamma}{r}}.$$

Expanding the sines and cosines in series, we get

$$\cos A = \frac{\left(1 - \frac{\alpha^2}{2r^2} + \dots\right) - \left(1 - \frac{\beta^2}{2r^2} + \dots\right)\left(1 - \frac{\gamma^2}{2r^2} + \dots\right)}{\left(\frac{\beta}{r} - \frac{1}{6}\frac{\beta^3}{r^3} + \dots\right)\left(\frac{\gamma}{r} - \frac{1}{6}\frac{\gamma^3}{r^3} + \dots\right)}$$

Hence, retaining terms involving only up to $\frac{1}{r^2}$ and taking r to be infinite, we have

$$\cos A = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma}$$

which is the expression for the cosine of an angle in terms of the sides in Plane Trigonometry.

Similarly for the sine formula we have

$$\frac{\sin A}{\sin B}=\frac{\sin a}{\sin b}=\frac{\sin\frac{\alpha}{r}}{\sin\frac{\beta}{r}}$$

which on expansion becomes

$$\frac{\frac{\alpha}{r}-\frac{1}{3!}\frac{\alpha^3}{r^3}+\ldots}{\frac{\beta}{r}-\frac{1}{3!}\frac{\beta^3}{r^3}+\ldots}=\frac{\alpha}{\beta}+\frac{\alpha(\beta^2-\alpha^2)}{6\beta r^2}+\ldots\ldots$$

Hence taking r to be infinite, we have

$$\frac{\sin A}{\sin B}=\frac{\alpha}{\beta},$$

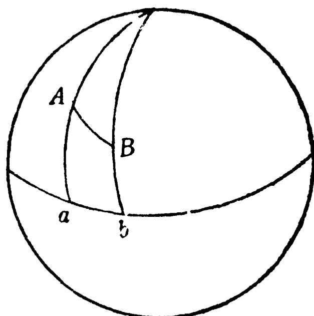
i.e., in a plane triangle the sines of the angles are proportional to the opposite sides.*

3.7. Distance between any two places on Earth's surface.

Let A and B be two places on Earth's surface and let their latitudes and longitudes be l_1, l_2 and λ_1, λ_2 respectively. Take P as the pole of the

* This formula is implied in *Khandakhādyaka*, VI, 1, by **Brahmagupta.** See the English edition by P. C. Sen Gupta, p. 115.

Equator and draw two secondaries to it through A and B respectively meeting it at a and b respectively.



Then $Aa = l_1$, $Bb = l_2$ and $ab = \lambda_2 - \lambda_1$.

Now from the triangle PAB, we have

$$\cos AB = \cos PA \cos PB + \sin PA \sin PB \cos APB$$

or denoting the arc AB by δ, we have

$$\cos \delta = \sin l_1 \sin l_2 + \cos l_1 \cos l_2 \cos (\lambda_1 - \lambda_2) \quad \dots \quad (1)$$

This formula can be put in another form, from which δ can be obtained when A and B are very close to each other. For we have

$$\cos \delta = \sin l_1 \sin l_2 \{\cos^2 \tfrac{1}{2}(\lambda_1 - \lambda_2) + \sin^2 \tfrac{1}{2}(\lambda_1 - \lambda_2)\}$$
$$+ \cos l_1 \cos l_2 \{\cos^2 \tfrac{1}{2}(\lambda_1 - \lambda_2) - \sin^2 \tfrac{1}{2}(\lambda_1 - \lambda_2)\}$$

$$= \cos (l_1 - l_2) \cos^2 \tfrac{1}{2}(\lambda_1 - \lambda_2) - \cos (l_1 + l_2)$$
$$\sin^2 \tfrac{1}{2}(\lambda_1 - \lambda_2).$$

Subtracting this from

$$1=\cos^2 \tfrac{1}{2}(\lambda_1-\lambda_2)+\sin^2 \tfrac{1}{2}(\lambda_1-\lambda_2)$$

we get

$$\sin^2 \frac{\delta}{2}=\cos^2 \tfrac{1}{2}(\lambda_1-\lambda_2) \sin^2 \tfrac{1}{2}(l_1-l_2)$$
$$+\sin^2 \tfrac{1}{2}(\lambda_1-\lambda_2) \cos^2 \tfrac{1}{2}(l_1+l_2). \quad \dots \quad (2)$$

Hence when A and B are very close together, the approximate value of δ is given by

$$\delta^2=(l_1-l_2)^2+(\lambda_1-\lambda_2)^2 \cos^2 \tfrac{1}{2}(l_1+l_2). \quad \dots \quad (3)$$

Examples worked out

Ex. 1. If D be any point in the side BC of a triangle ABC, shew that

$$\cos AD \sin BC=\cos AB \sin CD+\cos AC \sin BD.$$

We have $\cos ADB=\dfrac{\cos AB-\cos AD \cos BD}{\sin AD \sin BD}$

and $\cos ADC=\dfrac{\cos AC-\cos AD \cos CD}{\sin AD \sin CD}$

But $\cos ADB=-\cos ADC.$

Hence $\cos AB \sin CD+\cos AC \sin BD$

$$=\cos AD(\sin BD \cos CD+\cos BD \sin CD)$$
$$=\cos AD \sin BC.$$

Ex. 2. In any triangle, shew that

$$\frac{\sin (A+B)}{\sin C}=\frac{\cos a+\cos b}{1+\cos c},$$

We have $$\frac{\sin(A+B)}{\sin C}=\frac{\sin A\cos B+\cos A\sin B}{\sin C}$$

$$=\frac{\cos b-\cos a\cos c}{\sin^2 c}+\frac{\cos a-\cos b\cos c}{\sin^2 c},$$

by Arts. 31 & 3.4

$$=\frac{\cos a+\cos b}{1+\cos c}.$$

Ex. 3. If α, β and γ be the arcs joining the middle points of the sides of a spherical triangle ABC, shew that

$$\frac{\cos\alpha}{\cos\frac{a}{2}}=\frac{\cos\beta}{\cos\frac{b}{2}}=\frac{\cos\gamma}{\cos\frac{c}{2}}=\frac{1+\cos a+\cos b+\cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}}.$$

The arc α joins the middle points of b and c. Hence we have

$$\cos\alpha=\cos\frac{b}{2}\cos\frac{c}{2}+\sin\frac{b}{2}\sin\frac{c}{2}\cos A$$

$$=\cos\frac{b}{2}\cos\frac{c}{2}+\sin\frac{b}{2}\sin\frac{c}{2}\cdot\frac{\cos a-\cos b\cos c}{\sin b\sin c},$$ by Art. 3.1

$$=\frac{(1+\cos b)(1+\cos c)+\cos a-\cos b\cos c}{4\cos\frac{b}{2}\cos\frac{c}{2}}$$

$$=\frac{1+\cos a+\cos b+\cos c}{4\cos\frac{b}{2}\cos\frac{c}{2}}.$$

$$\therefore\quad\frac{\cos\alpha}{\cos\frac{a}{2}}=\frac{1+\cos a+\cos b+\cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}}.$$

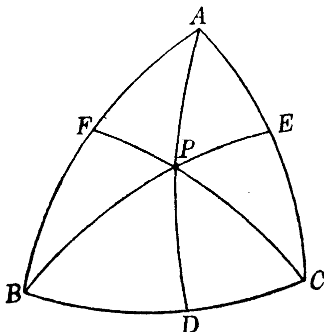
Similar expressions are obtained for $\cos\beta$ and $\cos\gamma$. Hence the result.

Ex. 4. In a spherical triangle ABC, great circular arcs α, β and γ are drawn from the vertices A, B and C perpendicular to the opposite sides and terminated by them. Shew that

$$\sin a \sin \alpha = \sin b \sin \beta = \sin c \sin \gamma$$

$$= \sqrt{(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c)}$$

(*C.U., M.A. & M.Sc., 1932.*)



Let α, β and γ meet the opposite sides in D, E and F respectively. Then from the triangle ABD, we have

$$\sin \alpha = \sin c \sin B, \text{ by Art. (3.4).}$$

$$\therefore \quad \sin a \sin \alpha = \sin a \sin B \sin c.$$

Similarly from the triangles BEC and BFC, we have

$$\sin \beta = \sin a \sin C \quad \text{and} \quad \sin \gamma = \sin a \sin B.$$

Hence $\sin a \sin \alpha = \sin b \sin \beta = \sin C \sin \gamma = \sin a \sin b \sin C$,

$$= \sin a \sin b \frac{2n}{\sin a \sin b} = 2n$$

$$= \sqrt{(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c)}.$$

Definition. The length of the great circular arc, drawn from the vertex of a spherical triangle perpendicular on the opposite side and terminated by it, is called an *Altitude* of the triangle. Thus in the above example α, β and γ are the three altitudes of the triangle ABC.

The above example shews that

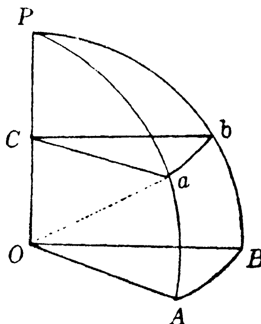
The product of the sine of a side and the sine of the corresponding altitude has the same value, whichever side be taken.

Ex. 5. Two ports are in the same parallel of latitude, their common latitude being l and their difference of longitude 2λ; shew that the saving of distance in sailing from one to the other on the great circle, instead of sailing due east or west is

$$2r\{\lambda \cos l - \sin^{-1}(\sin\lambda\cos l)\},$$

λ being expressed in circular measure, and r being the radius of the earth.

(*C. U., M. A. & M.Sc. 1931.*)



Let a and b be the two ports. Through the pole P of the small circle ab draw great circular arcs PaA and PbB to meet the great circle of which P is the pole (the equator) at A and B. Then AB is the difference of longitude.

Let the arcual distance ab along the small circle be s and along a great circle be d, so that their respective circular measures are $\frac{s}{r}$ and $\frac{d}{r}$.

Now by Art. 1.14, $\frac{\text{arc } ab}{\text{arc } AB} = \cos AOa = \cos l$.

$$\therefore \quad \frac{s}{r} = 2\lambda \cos l.$$

Again $\cos \frac{d}{r} = \cos^2 \left(\frac{\pi}{2} - l\right) + \sin^2 \left(\frac{\pi}{2} - l\right) \cos 2\lambda,$

$$= \sin^2 l + \cos^2 l \cos 2\lambda.$$

Hence $2 \sin^2 \frac{d}{2r} = \cos^2 l - \cos^2 l \cos 2\lambda = 2 \cos^2 l \sin^2 \lambda.$

$$\therefore \quad \sin \frac{d}{2r} = \cos l \sin \lambda$$

or $$\frac{d}{2r} = \sin^{-1} (\cos l \sin \lambda).$$

Hence the required saving of distance $= s - d$

$$= 2r \lambda \cos l - 2 r \sin^{-1} (\sin \lambda \cos l)$$

$$= 2r \{\lambda \cos l - \sin^{-1} (\sin \lambda \cos l)\}.$$

Examples

1. If $A = a$, shew that B and b are equal or supplemental, as also C and c.

2. The base BC of the triangle ABC is bisected at D. Shew that

(i) $\cos AB + \cos AC = 2 \cos AD . \cos BD$.

(ii) $\sin BAD : \sin CAD = \sin b : \sin c$.

3. In an equilateral triangle, shew that

(i) $\sec A = 1 + \sec a$,

(ii) $2 \cos \frac{a}{2} \sin \frac{A}{2} = 1$.

(iii) $\tan^2 \frac{a}{2} = 1 - 2 \cos A$.

4. If an angle of a triangle be equal or supplemental to the opposite side, shew that

$$1-\sec^2 a-\sec^2 b-\sec^2 c+2\sec a\sec b\sec c=0$$

5. If δ be the length of the arc joining the middle point of the side AB with the vertex C, shew that

$$\cos\delta=\frac{\cos a+\cos b}{2\cos\frac{1}{2}c}$$

6. The base BC of the triangle ABC is bisected at X, and a point Y is taken on BC such that the $\angle BAX=\angle CAY$. Shew that

$$\sin BY:\sin CY=\sin^2 c:\sin^2 b.$$

7. In a triangle ABC, α, β, γ are drawn perpendiculars from the vertices A, B, C on the opposite sides. Shew that

(*i*) $\sin a\cos\alpha=\surd\{\cos^2 b+\cos^2 c-2\cos a\cos b\cos c\}$,

(*ii*) $\sin b\cos\beta=\surd\{\cos^2 a+\cos^2 c-2\cos a\cos b\cos c\}$,

(*iii*) $\sin c\cos\gamma=\surd\{\cos^2 a+\cos^2 b-2\cos a\cos b\cos c\}$.

8. Prove that

$$8n^3=\sin^2 a\sin^2 b\sin^2 c\sin A\sin B\sin C.$$

9. In any triangle, shew that

$$\frac{\sin(A-B)}{\sin C}=\frac{\cos b-\cos a}{1-\cos c}.$$

10. If α' be the arc joining the middle points of the sides $A'B$ and $A'C$ of the colunar triangle of ABC, shew that

$$\cos\alpha'=\frac{1+\cos a-\cos b-\cos c}{4\sin\frac{1}{2}b\sin\frac{1}{2}c}.$$

11. If α, β and γ be the arcs joining the middle points of the sides of a spherical triangle, shew that when one of them is a quadrant, the other two are also quadrants.

12. A port is in latitude l (North) and longitude λ (East). Shew that the longitudes of places on the Equator distant δ from the port are

$$\lambda \pm \cos^{-1}\left(\frac{\cos \delta}{\cos l}\right).$$

(*Science and Art Exam. Papers.*)

13. Two places on the Earth's surface are distant, one θ from the Pole and the other θ from the Equator, and their difference of longitude is ϕ; shew that the angular distance between them is

$$\cos^{-1}(\sin 2\,\theta \cos^2 \tfrac{1}{2}\phi).$$

(*Science and Art Exam. Papers.*)

3.8. Expressions for the sine, cosine and tangent of half an angle.

We know that

$$\cos A = 1 - 2\sin^2 \frac{A}{2}.$$

Hence, $\sin^2 \dfrac{A}{2} = \dfrac{1-\cos A}{2}$

$$= \tfrac{1}{2}\left\{1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}\right\}$$

by Art. 3.1

$$= \tfrac{1}{2}\left\{\frac{\cos(b-c) - \cos a}{\sin b \sin c}\right\}$$

$$= \frac{\sin \tfrac{1}{2}(a+b-c)\ \sin \tfrac{1}{2}(a-b+c)}{\sin b \sin c}$$

Put $2s=a+b+c$, then s denotes the half of the sum of the sides of triangle, and

$$a+b-c=2(s-c),\quad a-b+c=2(s-b)$$

so that

$$\sin^2\frac{A}{2}=\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}$$

Hence, $$\sin\frac{A}{2}=\sqrt{\left\{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}\right\}}.^{*}$$

Similarly, $$\sin\frac{B}{2}=\sqrt{\left\{\frac{\sin(s-c)\sin(s-a)}{\sin c\sin a}\right\}},$$

and $$\sin\frac{C}{2}=\sqrt{\left\{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}\right\}}.$$

Again,

$$\cos^2\frac{A}{2}=\frac{1+\cos A}{2}$$

$$=\tfrac{1}{2}\left\{1+\frac{\cos a-\cos b\cos c}{\sin b\sin c}\right\}$$

$$=\tfrac{1}{2}\left\{\frac{\cos a-\cos(b+c)}{\sin b\sin c}\right.$$

$$=\frac{\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(b+c-a)}{\sin b\sin c}$$

$$=\frac{\sin s\sin(s-a)}{\sin b\sin c}.$$

* Obtained by **Euler** (1707-1783) in 1753.

Hence $\cos\frac{A}{2} = \sqrt{\left\{\frac{\sin s \sin(s-a)}{\sin b \sin c}\right\}}$.*

Similarly, $\cos\frac{B}{2} = \sqrt{\left\{\frac{\sin s \sin(s-b)}{\sin c \sin a}\right\}}$,

and $\cos\frac{C}{2} = \sqrt{\left\{\frac{\sin s \sin(s-c)}{\sin a \sin b}\right\}}$.

From the above results, we get

$$\tan\frac{A}{2} = \sqrt{\left\{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}\right\}}\ *$$

$$\tan\frac{B}{2} = \sqrt{\left\{\frac{\sin(s-c)\sin(s-a)}{\sin s \sin(s-b)}\right\}}$$

and $\tan\frac{C}{2} = \sqrt{\left\{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}\right\}}$.

The radicals in the results of this article must be taken with positive signs, since the half angles are each less than a right angle and hence their sines, cosines and tangents are all positive.

3.9. Again since $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$,

we have

$$\sin A = \frac{2}{\sin b \sin c}\{\sin s \sin(s-a)\sin(s-b)\sin(s-c)\}^{\frac{1}{2}}.$$

* **Euler,** *l. c.*

Comparing it with the expression for sin A as given in Art. 3.3 we get

$$n^2 = \sin s . \sin (s-a) \sin (s-b) \sin (s-c)$$

$$= \frac{1}{4}\{1-\cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c\}.*$$

3.10. Analogous formulae in Plane Trigonometry.

Taking α, β, γ to be the lengths of the sides of the spherical triangle, we have $\frac{\alpha}{r}$, $\frac{\beta}{r}$ and $\frac{\gamma}{r}$ as their circular measures. Then

$$\cos \frac{A}{2} = \left\{\frac{\sin s \sin (s-a)}{\sin b \sin c}\right\}^{\frac{1}{2}} = \left\{\frac{\sin \frac{s'}{r} \sin \left(\frac{s'-\alpha}{r}\right)}{\sin \frac{\beta}{r} \sin \frac{\gamma}{r}}\right\}^{\frac{1}{2}}$$

where $2s' = \alpha + \beta + \gamma$.

Hence on expanding the sines and cosines, we have

$$\cos \frac{A}{2} = \left[\frac{\left(\frac{s'}{r} - \frac{1}{6}\frac{s'^3}{r^3} + \ldots\right)\left\{\frac{s'-\alpha}{r} - \frac{1}{6}\frac{(s'-\alpha)^3}{r^3} + \ldots\right\}}{\left(\frac{\beta}{r} - \frac{1}{6}\frac{\beta^3}{r^3} + \ldots\right)\left(\frac{\gamma}{r} - \frac{1}{6}\frac{\gamma^3}{r^3} + \ldots\right)}\right]^{\frac{1}{2}}.$$

Thus retaining only up to the second power of r and taking r to be infinite, we get

$$\cos \frac{A}{2} = \sqrt{\frac{s'(s'-\alpha)}{\beta\gamma}}$$

* These expressions for n are given by **Euler** in *Novi Commentarii Petropolitana*, Vol. IV, p. 158.

for the relation for a plane triangle.

Similarly, $\sin \frac{A}{2} = \sqrt{\frac{(s'-\beta)(s'-\gamma)}{\beta\gamma}}$,

and $\tan \frac{A}{2} = \sqrt{\frac{(s'-\beta)(s'-\gamma)}{s'(s'-\alpha)}}$.

Again from the relation

$$\sin A = \frac{2n}{\sin b \sin c} = \frac{2\{\sin s \sin (s-a) \sin (s-b) \sin (s-c)\}^{\frac{1}{2}}}{\sin b \sin c}$$

we get

$$\sin A = \frac{2\{s'(s'-\alpha)(s'-\beta)(s'-\gamma)\}^{\frac{1}{2}}}{\beta\gamma}$$

so that the area of the plane triangle ABC is

$$\triangle = \{s'(s'-\alpha)(s'-\beta)(s'-\gamma)\}^{\frac{1}{2}}.$$

This form is due to **Heron** of Alexandria (50 A.D.).

EXAMPLES

In any spherical triangle, shew that

1. $\tan \frac{1}{2}A \tan \frac{1}{2}B = \frac{\sin (s-c)}{\sin s}$.

2. $\cot \frac{1}{2}A : \cot \frac{1}{2}B : \cot \frac{1}{2}C = \sin(s-a) : \sin(s-b) : \sin (s-c)$.

3. $\sin s = \frac{\cos \frac{1}{2}B \cos \frac{1}{2}C}{\sin \frac{1}{2}A} \sin a$.

4. $\sin (s-a) = \frac{\sin \frac{1}{2}B \sin \frac{1}{2}C}{\sin \frac{1}{2}A} \sin a$,

5. $\sin s \sin a \sin b \sin c \sin \frac{1}{2} A \sin \frac{1}{2}B \sin \frac{1}{2}C = n^2$,

6. $\operatorname{cosec} \frac{1}{2}A = \frac{\cos \frac{1}{2}B}{\cos \frac{1}{2}C} \cdot \frac{\sin c}{\sin (s-b)}$.

3.11. Expression for the cosine of a side in terms of sines and cosines of the angles.

Let a', b' and c' be the sides and A', B' and C' the angles of the polar triangle of ABC. Then by Art. 3.1 we have

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

Substituting the values of a', b', c' and A' from Art. 2.6 we have

$$\cos(\pi - A) = \cos(\pi - B)\cos(\pi - C) + \sin(\pi - B)\sin(\pi - C)\cos(\pi - a),$$

that is, $\cos A = -\cos B \cos C + \sin B \sin C \cos a$.

Similarly, $\cos B = -\cos C \cos A + \sin C \sin A \cos b$,

and $\cos C = -\cos A \cos B + \sin A \sin B \cos c$.

These * can also be written as

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\cos b = \frac{\cos B + \cos C \cos A}{\sin C \sin A},$$

and $$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

* These formulae are due to the French Mathematician **Vieta** (1540-1603) who published them in the eighth book of his *Variorum de rebus mathematicis responsorum* in 1595.

3.12. Analogous formula for plane tri.ngle.

When r the radius of the sphere is taken to be infinite, we have

$$\cos a = \cos \frac{a}{r} = 1.$$

Hence the formula

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

becomes $\cos A = -\cos B \cos C + \sin B \sin C$

so that $B + C = \pi - A$ or $A + B + C = \pi$,

showing that the three angles of a plane triangle are together equal to two right angles.

3.13. Expressions for the sine, cosine and tangent of half of a side in terms of sines and cosines of the angles.

We have $\sin^2 \frac{a}{2} = \frac{1 - \cos a}{2}$

$$= \frac{1}{2}\left\{1 - \frac{\cos A + \cos B \cos C}{\sin B \sin C}\right\}$$

$$= \frac{1}{2}\left\{-\frac{\cos A + \cos (B + C)}{\sin B \sin C}\right\}$$

$$= -\frac{\cos \frac{1}{2}(A + B + C) \cos \frac{1}{2}(B + C - A)}{\sin B \sin C}.$$

Putting $2S = A + B + C$, we have

$$\sin\frac{a}{2} = \sqrt{\left\{-\frac{\cos S \cos (S-A)}{\sin B \sin C}\right\}}$$

with similar expressions for $\sin\frac{1}{2}b$ and $\sin\frac{1}{2}c$.

Again $\cos^2\frac{a}{2} = \frac{1+\cos a}{2}$

$$= \tfrac{1}{2}\left\{1 + \frac{\cos A + \cos B \cos C}{\sin B \sin C}\right\}$$

$$= \tfrac{1}{2}\left\{\frac{\cos A + \cos (B-C)}{\sin B \sin C}\right\}$$

$$= \frac{\cos\frac{1}{2}(A-B+C) \cos\frac{1}{2}(A+B-C)}{\sin B \sin C}$$

$$= \frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}.$$

Hence $\cos\frac{a}{2} = \sqrt{\left\{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}\right\}}$.

Also $\tan\frac{a}{2} = \sqrt{\left\{-\frac{\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}\right\}}$.

The radicals must be taken with positive signs since $\frac{1}{2}a$ is less than a right angle.

It is to be noted here that the value of S lies between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$. Hence the value of $\cos S$ is negative and the values of $S-A$, $S-B$ and $S-C$ lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ (Arts. 2.9 and 2.10) so that their cosines are positive. Hence the expressions within brackets are positive so that the values of $\sin \frac{1}{2}a$, $\cos \frac{1}{2}a$ and $\tan \frac{1}{2}a$ are all real and positive.

The above results could have been obtained from the results of Arts. 3.1 and 3.8 by changing the sides and angles into the supplements of angles and sides. They illustrate the proposition that if a theorem holds good between the sides and angles of a spherical triangle, the theorem will remain true when the sides and angles are changed into the supplements of the corresponding angles and sides respectively. (Art. 2.6, note.)

3.14. Expression for the sine of a side.

We have $\sin a = 2 \sin \frac{1}{2}a \cos \frac{1}{2}a$

$$= \frac{2}{\sin B \sin C} \left\{ -\cos S \cos(S-A)\cos(S-B)\cos(S-C) \right\}^{\frac{1}{2}}.$$

We shall use the symbol N to denote

$$\{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)\}^{\frac{1}{2}};$$

then $$\sin a = \frac{2N}{\sin B \sin C}.$$

Similarly, $\sin b = \dfrac{2N}{\sin C \sin A}$ and $\sin c = \dfrac{2N}{\sin A \sin B}$.

Thus

$$2N = \sin a \sin B \sin C = \sin A \sin b \sin C$$
$$= \sin A \sin B \sin c.$$

N is called the *Norm of the angles* * of the spherical triangle.

Examples worked out

Ex. 1. In any triangle shew that

$$\frac{\cos A + \cos B}{1 - \cos C} = \frac{\sin (a+b)}{\sin c}$$

We have $\cos A = -\cos B \cos C + \sin B \sin C \cos a$

and $\cos B = -\cos A \cos C + \sin A \sin C \cos b.$

Adding these we get

$$\cos A + \cos B = -\cos C (\cos A + \cos B)$$
$$+ \sin C (\sin B \cos a + \sin A \cos b),$$

whence, $(\cos A + \cos B)(1 + \cos C)$

$$= \sin^2 C . \frac{(\sin b \cos a + \sin a \cos b)}{\sin c}, \text{ by Art. 3.4.}$$

Thus $$\frac{\cos A + \cos B}{1 - \cos C} = \frac{\sin (a+b)}{\sin c}.$$

* Due to Professor **Neuberg.** Professor Von **Staudt** calls $2N$ *sine of the polar triangle.* Various expressions for N were given by **Lexell** in *Acta Petropolitana*, 1782, p. 49.

Ex. 2. If θ and θ' denote the angles which the internal and external bisectors of the angle C make with the side AB, shew that

$$\cos\theta = \frac{\cos A \sim \cos B}{2\cos\frac{1}{2}C}$$

and $$\cos\theta' = \frac{\cos A + \cos B}{2\sin\frac{1}{2}C}.$$

Let δ and δ' be the lengths of the internal and external bisectors of the angle C and let them meet AB at D and E respectively, making with it the angles θ and θ'. Then from the triangle ACD, we have by Art. 3.11

$$\cos\delta = \frac{\cos A + \cos\theta\cos\frac{1}{2}C}{\sin\theta\sin\frac{1}{2}C}.$$

Similarly from the triangle BCD, we have

$$\cos\delta = \frac{\cos B - \cos\theta\cos\frac{1}{2}C}{\sin\theta\sin\frac{1}{2}C}.$$

Equating these two values of $\cos\delta$, we have

$$\cos\theta = \frac{\cos A \sim \cos B}{2\cos\frac{1}{2}C}.$$

Again from the triangles AEC and BEC, we have

$$\cos\delta' = \frac{\cos A + \cos\theta'\cos\frac{1}{2}(\pi + C)}{\sin\theta'\sin\frac{1}{2}(\pi + C)}$$

$$= \frac{\cos(\pi - B) + \cos\theta'\cos\frac{1}{2}(\pi - C)}{\sin\theta'\sin\frac{1}{2}(\pi - C)},$$

whence, $$\cos\theta' = \frac{\cos A + \cos B}{2\sin\frac{1}{2}C}.$$

EXAMPLES

1. If the side BC of the triangle ABC be a quadrant, shew that

$$\cos A + \cos B \cos C = 0$$

2. In any triangle, shew that

$$\frac{\cos A - \cos B}{1 + \cos C} = \frac{\sin (b-a)}{\sin c}.$$

3. In any triangle, shew that

$$\Sigma \frac{\cos A + \cos B}{1 - \cos C} \sin (a-b) \sin c = 0,$$

and $$\Sigma \frac{\cos A - \cos B}{1 + \cos C} \sin (a+b) \sin c = 0.$$

4. In an equilateral triangle, shew that

$$\tan^2 \frac{a}{2} = 1 - 2 \cos A.$$

5. Shew that

$$4 N^2 = 1 - \cos^2 A - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C.$$

6. If α, β, γ be the arcs of great circles drawn from A, B, C perpendicular on the opposite sides and terminated by them, shew that

(*i*) $\sin A \sin \alpha = \sin B \sin \beta = \sin C \sin \gamma = 2N$,

(*ii*) $\sin a \sin \alpha = \sin b \sin \beta = \sin c \sin \gamma = 2n$.

7. Prove that

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = \frac{n}{N}.$$

8. Prove that

$$N = \frac{2n^2}{\sin a \sin b \sin c},$$

and $$n = \frac{2N^2}{\sin A \sin B \sin C}.$$

9. Shew that

$$2N = (\sin a \sin b \sin c \sin^2 A \sin^2 B \sin^2 C)^{\frac{1}{3}}.$$

10. Shew that

$$4nN = \sin a \sin b \sin c \sin A \sin B \sin C.$$

11. Shew that

$$\tan \frac{b}{2} \tan \frac{c}{2} = \frac{-\cos S}{\cos (S-A)}.$$

12. Shew that

$$\tan \frac{a}{2} : \tan \frac{b}{2} : \tan \frac{c}{2} = \cos(S-A) : \cos(S-B) : \cos(S-C).$$

13. Shew that

$$\frac{\sin^2 a + \sin^2 b + \sin^2 c}{\sin^2 A + \sin^2 B + \sin^2 C} = \frac{1-\cos a \cos b \cos c}{1+\cos A \cos B \cos C}.$$

(*Dublin University Examination Papers.*)

14. Shew that

$$\frac{\cos (S-A)}{\sin A} = \frac{\cos \frac{1}{2}b \cos \frac{1}{2}c}{\cos \frac{1}{2}a}.$$

15. Shew that

$$-\cos S = \frac{\sin \frac{1}{2}a \sin \frac{1}{2}b \sin C}{\cos \frac{1}{2}c}$$

$$= \frac{n}{2 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}.$$

16. Shew that

$$\cot \tfrac{1}{2}a \cos (S-A) = \cot \tfrac{1}{2}b \cos (S-B) = \cot \tfrac{1}{2}c \cos (S-C)$$

$$= -\cot \tfrac{1}{2}a \cot \tfrac{1}{2}b \cot \tfrac{1}{2}c \cos S = \frac{n}{2 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}.$$

3.15. Relations existing between two sides, the included angle and another angle.

Cotangent formulae. *In any spherical triangle,*

$$\cot a \sin b = \cot A \sin C + \cos b \cos C.$$

We have

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$= \cos b (\cos a \cos b + \sin a \sin b \cos C)$$

$$+ \frac{\sin b \sin a \sin C \cos A}{\sin A}$$

by substituting the values of $\sin c$ and $\cos c$.

Thus

$$\cos a (1 - \cos^2 b) = \sin a \sin b \cos b \cos C,$$
$$+ \sin a \sin b \sin C \cot A,$$

or, $\cos a \sin^2 b = \sin a \sin b (\cos b \cos C + \cot A \sin C)$,

i.e., $\cot a \sin b = \cot A \sin C + \cos b \cos C.$

By proceeding similarly we can get five other formulae, namely,

$$\cot b \sin a = \cot B \sin C + \cos a \cos C.$$

$$\cot b \sin c = \cot B \sin A + \cos c \cos A.$$

$$\cot c \sin b = \cot C \sin A + \cos b \cos A.$$

$$\cot c \sin a = \cot C \sin B + \cos a \cos B.$$

$$\cot a \sin c = \cot A \sin B + \cos c \cos B.$$

Of the four elements entering into any one of the formulae it will be noticed that one side lies between two angles and one angle is included by the two sides, and if we denote them by 1 and 2, and the remaining side and angle by 3 and 4 respectively, all the formulae * are expressed in the form

$$\cos 1 \cos 2 = \begin{vmatrix} \sin 1 & \sin 2 \\ \cot 4 & \cot 3 \end{vmatrix}$$

3.16. Napier's analogies.

We have

$$\tan \tfrac{1}{2}(A+B) \tan \tfrac{1}{2}C = \frac{\tan \frac{1}{2}A \tan \frac{1}{2}C + \tan \frac{1}{2}B \tan \frac{1}{2}C}{1 - \tan \frac{1}{2}A \tan \frac{1}{2}B}.$$

Substituting the values of tangents from Art. 3.8 we get

$$\tan \tfrac{1}{2}(A+B) \tan \tfrac{1}{2}C = \frac{\sin(s-a) + \sin(s-b)}{\sin s - \sin(s-c)}$$

$$= \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}.$$

Thus, $\tan \tfrac{1}{2}(A+B) \tan \tfrac{1}{2}C = \dfrac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}$. ... (1)

Similarly,

$$\tan \tfrac{1}{2}(A-B) \tan \tfrac{1}{2}C = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)}. \quad \ldots \quad (2)$$

* Dr. Leathem states the formula in the form

(cosine of inner side) (cosine of inner angle)

= *(sine of inner side) (cotangent of other side)*

− *(sine of inner angle) (cotangent of other angle)*.

Again by substituting the elements of the polar triangle in (1) and (2), or proceeding as in (1) and (2) with tangents of half sides (Art. 3.13) we get

$$\frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}c}=\frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}. \quad \dots \quad (3)$$

and

$$\frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2}c}=\frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}. \quad \dots \quad (4)$$

The above four formulae are known as Napier's analogies.*

As a, b and C are less than π (Art. 2.2), $\cos \frac{1}{2}(a-b)$ and $\tan \frac{1}{2}C$ are essentially positive. Hence in (1) $\tan \frac{1}{2}(A+B)$ and $\cos \frac{1}{2}(a+b)$ must have the same sign. Therefore $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ must either be both greater than $\frac{1}{2}\pi$ or both less than $\frac{1}{2}\pi$, *i.e.*, $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ *are of the same affection.* The same result follows from (3) also.

3.17. Delambre's analogies.

We have $\sin \frac{1}{2}(A+B)=\sin \frac{1}{2}A \cos \frac{1}{2}B+\cos \frac{1}{2}A \sin \frac{1}{2}B.$

Substituting for $\sin \frac{1}{2}A$, $\cos \frac{1}{2}B$, etc., their equivalents from Art. 3.8, we get

$$\sin \frac{1}{2}(A+B)=\frac{\sin (s-b)+\sin (s-a)}{\sin c}\sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}}$$

$$=\frac{\sin (s-b)+\sin (s-a)}{\sin c}\cos \frac{1}{2}C$$

$$=\frac{2 \sin \frac{1}{2}c \cos \frac{1}{2}(a-b)}{\sin c}\cos \frac{1}{2}C.$$

* **Napier** (1550-1617) discovered these analogies and published them in his *Mirifici Logarithmorum Canonis Descriptio* in 1614.

Hence $$\frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}C}=\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}c}. \quad \dots \quad (1)$$

Similarly,

$$\frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C}=\frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}c}, \quad \dots \quad (2)$$

$$\frac{\cos \frac{1}{2}(A+B)}{\sin \frac{1}{2}C}=\frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}c}, \quad \dots \quad (3)$$

and $$\frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C}=\frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}c}. \quad \dots \quad (4)$$

The above four formulae are known as Delambre's analogies and were obtained by him in 1807, though published afterwards in *Connaissance des Tems.*, 1809, p. 443. Sometimes they are improperly called *Gauss's Theorems.**

3.18. Napier's analogies can easily be obtained from those of Delambre. Thus dividing (1) by (3) and (2) by (4) we get the first two analogies of Napier. Similarly dividing (4) by (3) and (2) by (1) we get the other two analogies of Napier. Delambre's analogies

* According to Professor **Simon Newcomb** (1835-1909) these analogies were first published anonymously by **Delambre** (1749-1822) although **Gauss** (1777-1855) was the first to use them in Spherical Astronomy. **Gauss** published them in *Theoria motus corporum coelestium* in 1809 and **Mollweide** in **Zach's** *Monatliche Correspondenz* in 1808.

also may be obtained from those of Napier. Thus squaring the first analogy of Napier, we have

$$\tan^2 \tfrac{1}{2}(A+B) = \frac{\cos^2 \tfrac{1}{2}(a-b)}{\cos^2 \tfrac{1}{2}(a+b)} \cot^2 \tfrac{1}{2}C.$$

Adding 1 to both sides, we get

$$\sec^2 \tfrac{1}{2}(A+B)$$

$$= \frac{\cos^2 \tfrac{1}{2}(a-b)\cos^2 \tfrac{1}{2}C + \cos^2 \tfrac{1}{2}(a+b)\sin^2 \tfrac{1}{2}C}{\cos^2 \tfrac{1}{2}(a+b)\sin^2 \tfrac{1}{2}C}$$

$$= \frac{\tfrac{1}{2}\{1+\cos(a-b)\}\cos^2 \tfrac{1}{2}C + \tfrac{1}{2}\{1+\cos(a+b)\}\sin^2 \tfrac{1}{2}C}{\cos^2 \tfrac{1}{2}(a+b)\sin^2 \tfrac{1}{2}C}$$

$$= \frac{\tfrac{1}{2}(1+\cos a\cos b+\sin a\sin b\cos C)}{\cos^2 \tfrac{1}{2}(a+b)\sin^2 \tfrac{1}{2}C}$$

$$= \frac{\tfrac{1}{2}(1+\cos c)}{\cos^2 \tfrac{1}{2}(a+b)\sin^2 \tfrac{1}{2}C} = \frac{\cos^2 \tfrac{1}{2}c}{\cos^2 \tfrac{1}{2}(a+b)\sin^2 \tfrac{1}{2}C}$$

whence $$\frac{\cos \tfrac{1}{2}(A+B)}{\sin \tfrac{1}{2}C} = \frac{\cos \tfrac{1}{2}(a+b)}{\cos \tfrac{1}{2}c},$$

which is the third analogy of Delambre. Other analogies can also be obtained similarly.

3.19. Deduction of the analogies of Napier and Delambre.

We have from Art. 3.4

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c},$$

so that $$\frac{\sin A \pm \sin B}{\sin a \pm \sin b} = \frac{\sin C}{\sin c}. \qquad \dots \quad (1)$$

Again we have from Ex. 2, p. 39

$$\frac{\sin (A+B)}{\sin C}=\frac{\cos a+\cos b}{1+\cos c}. \qquad \ldots \quad (2)$$

And from the polar triangle of ABC, we get (Ex. 1, p. 53)

$$\frac{(\sin a+b)}{\sin c}=\frac{\cos A+\cos B}{1-\cos C}. \qquad \ldots \quad (3)$$

Hence

$$\frac{\sin A+\sin B}{\cos A+\cos B}=\frac{\sin a+\sin b}{\sin c}\cdot\frac{\sin C}{1-\cos C}\cdot\frac{\sin c}{\sin (a+b)},$$

or $$\tan \tfrac{1}{2}(A+B)=\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}C,$$

which is Napier's first analogy.

Again

$$\frac{\sin a+\sin b}{\cos a+\cos b}=\frac{\sin A+\sin B}{\sin C}\cdot\frac{\sin c}{1+\cos c}\cdot\frac{\sin C}{\sin (A+B)}$$

or $$\tan \tfrac{1}{2}(a+b)=\frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2}c,$$

which is the third analogy of Napier.

On taking the negative sign in (1) the other two analogies are obtained in a similar manner.

Next consider the colunar triangle $A''BC$ where A'' is the point diametrically opposite to A. For this triangle A and a are unaltered and the other

parts are changed into their supplements and (2) becomes (Ex. 9, p. 43).

$$\frac{\sin (A-B)}{\sin C}=\frac{\cos b-\cos a}{1-\cos c}, \qquad \ldots \quad (4)$$

and from the polar triangle of $A''BC$, we get (Ex. 2, p. 55)

$$\frac{\sin (a-b)}{\sin c}=\frac{\cos B-\cos A}{1+\cos C}. \qquad \ldots \quad (5)$$

Multiplying (1) by (5) we get

$$\frac{\sin A+\sin B}{\sin C}\cdot\frac{\cos B-\cos A}{1+\cos C}=\frac{\sin a+\sin b}{\sin c}\cdot\frac{\sin(a-b)}{\sin c},$$

or, $$\frac{\sin^2 \frac{1}{2}(A+B) \sin (A-B)}{\cos^2 \frac{1}{2}C \sin C}=\frac{\sin a+\sin b}{\sin^2 c}\cdot \sin (a-b),$$

or, $$\frac{\sin^2 \frac{1}{2}(A+B)}{\cos^2 \frac{1}{2}C}=\frac{\cos^2 \frac{1}{2}(a-b)}{\cos^2 \frac{1}{2}c}, \quad \text{by (4)},$$

which is the first analogy of Delambre.

Similarly multiplying (1) by (3) and dividing by (2) we get

$$\frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C}=\frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}c},$$

which is the fourth analogy of Delambre.

On taking the negative sign in (1) and multiplying it by (3) and (5) respectively, we get the remaining two analogies of Delambre.

Examples worked out

Ex. 1. If a spherical triangle is equal and similar to its polar triangle, shew that

(1) $\sec a = \sec b \sec c + \tan b \tan c$,

(2) $\sec^2 A + \sec^2 B + \sec^2 C + 2 \sec A \sec B \sec C = 1$,

(*Science and Art Exam. Papers.*)

(1) We have $\cos a = \cos b \cos c + \sin b \sin c \cos A$, by Art. 3.1

$$= \cos b \cos c + \sin b \sin c \cos (\pi - a)$$

$$= \cos b \cos c - \sin b \sin c \cos a$$

for $A = A' = \pi - a$.

Dividing both sides by $\cos a \cos b \cos c$ and transposing we get

$$\sec a = \sec b \sec c + \tan b \tan c.$$

(2) We have $\cos a = \cos b \cos c + \sin b \sin c \cos A$,

or $\cos (\pi - A) = \cos (\pi - B) \cos (\pi - C) + \sin (\pi - B) \sin (\pi - C) \cos A$,

for $a = \pi - A' = \pi - A$, etc.

Hence $-\cos A = \cos B \cos C + \sin B \sin C \cos A$,

or, $-\sec B \sec C = \sec A + \tan B \tan C$,

or, $\sec^2 A + \sec^2 B \sec^2 C + 2 \sec A \sec B \sec C = \tan^2 B \tan^2 C$

$$= \sec^2 B \sec^2 C - \sec^2 B - \sec^2 C + 1,$$

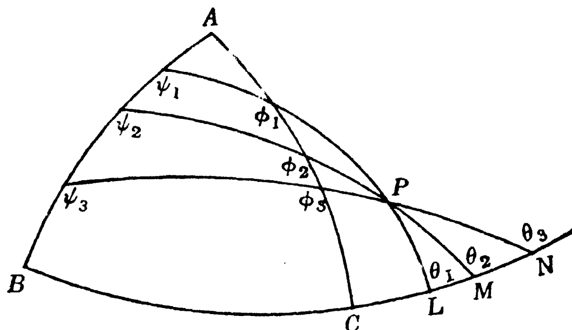
whence $\sec^2 A + \sec^2 B + \sec^2 C + 2 \sec A \sec B \sec C = 1$.

Ex. 2. Three great circles are drawn through a point P on the surface of a sphere, cutting the sides of the spherical triangle ABC and making with them the angles θ_1, ϕ_1, ψ_1; θ_2, ϕ_2, ψ_2 and θ_3, ϕ_3, ψ_3 respectively. Shew that

$$\begin{vmatrix} \cos \theta_1 & \cos \phi_1 & \cos \psi_1 \\ \cos \theta_2 & \cos \phi_2 & \cos \psi_2 \\ \cos \theta_3 & \cos \phi_3 & \cos \psi_3 \end{vmatrix} = 0.$$

Let the three arcs cut the side a at the points L, M and N, and let PL and PN make the angles α and β with PM.

Then from the triangle PLM, we have by Art. 3.11



$$\cos PM = \frac{\cos\theta_1 + \cos a \cos PML}{\sin a \sin PML} = \frac{\cos\theta_1 - \cos a \cos\theta_2}{\sin a \sin\theta_2}$$

Again from the triangle PMN, we have

$$\cos PM = \frac{-\cos\theta_3 + \cos\beta\cos\theta_2}{\sin\beta\sin\theta_2}.$$

Hence $\quad (\cos\theta_1 - \cos\alpha\cos\theta_2)\sin\beta = \sin\alpha(-\cos\theta_3 + \cos\beta\cos\theta_2)$

or, $\quad \cos\theta_1 \sin\beta + \cos\theta_3 \sin\alpha = \cos\theta_2 \sin(\alpha+\beta).$

Similarly, $\quad \cos\phi_1 \sin\beta + \cos\phi_3 \sin\alpha = \cos\phi_2 \sin(\alpha+\beta),$

and $\quad \cos\psi_1 \sin\beta + \cos\psi_3 \sin\alpha = \cos\psi_2 \sin(\alpha+\beta).$

Hence eliminating $\sin\alpha$, $\sin\beta$ and $\sin(\alpha+\beta)$ from the three equations, we get

$$\begin{vmatrix} \cos\theta_1 & \cos\phi_1 & \cos\psi_1 \\ \cos\theta_2 & \cos\phi_2 & \cos\psi_2 \\ \cos\theta_3 & \cos\phi_3 & \cos\psi_3 \end{vmatrix} = 0.$$

Ex. 3. If two great circular arcs are drawn from the vertex C of a spherical triangle ABC, one perpendicular on AB and the other bisecting the angle C, and ϕ be the angle between them, shew that

$$\tan\phi = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\tan\tfrac{1}{2}(A-B).$$

(*Dublin Univ. Exam. Papers.*)

Let the perpendicular and the bisector meet AB at D and E respectively.

From the triangle CAD, we have by Art. 3.11

$$\cos CD = \frac{\cos A}{\sin(\frac{1}{2}C-\phi)},$$

and from the triangle CBD, we get

$$\cos CD = \frac{\cos B}{\sin(\frac{1}{2}C+\phi)}.$$

Thus $$\frac{\cos A}{\cos B} = \frac{\sin(\frac{1}{2}C-\phi)}{\sin(\frac{1}{2}C+\phi)}.$$

Hence $$\frac{\cos A - \cos B}{\cos A + \cos B} = \frac{\sin(\frac{1}{2}C-\phi) - \sin(\frac{1}{2}C+\phi)}{\sin(\frac{1}{2}C-\phi) + \sin(\frac{1}{2}C+\phi)},$$

or, $$\tan\tfrac{1}{2}(A+B)\tan\tfrac{1}{2}(A-B) = \tan\phi\cot\tfrac{1}{2}C.$$

Hence substituting the value of $\tan\frac{1}{2}(A+B)$ from Napier's first analogy (Art. 3.16) we have

$$\tan\phi = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\tan\tfrac{1}{2}(A-B).$$

Note.—Substituting the value of $\tan\frac{1}{2}(A-B)$ from Napier's second analogy we get

$$\tan\phi = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\tan\tfrac{1}{2}(A+B).$$

Ex. 4. If δ be the length of the arc through the vertex of an isosceles triangle, dividing the base into segments α and β, shew that

$$\tan \tfrac{1}{2}\alpha \tan \tfrac{1}{2}\beta = \tan \tfrac{1}{2}(a+\delta) \tan \tfrac{1}{2}(a-\delta),$$

where a is one of the equal sides of the triangle.

(*C.U., M.A. & M.Sc.*, **1934.**)

Let ABC be an isosceles triangle having $AC = BC$, and let δ meet the base AB at D. Then from the triangle ADC, we have by Napier's third analogy (Art. 3.16)

$$\tan \tfrac{1}{2}(a+\delta) = \frac{\cos \tfrac{1}{2}(D-A)}{\cos \tfrac{1}{2}(D+A)} \tan \tfrac{1}{2}\alpha,$$

where D represents the angle ADC.

Again from the triangle BDC, we have by Napier's fourth analogy

$$\tan \tfrac{1}{2}(a-\delta) = \frac{\sin \tfrac{1}{2}(\pi - D - B)}{\sin \tfrac{1}{2}(\pi - D + B)} \tan \tfrac{1}{2}\beta$$

$$= \frac{\cos \tfrac{1}{2}(D+A)}{\cos \tfrac{1}{2}(D-A)} \tan \tfrac{1}{2}\beta.$$

Hence multiplying, we get

$$\tan \tfrac{1}{2}(a+\delta) \tan \tfrac{1}{2}(a-\delta) = \tan \tfrac{1}{2}\alpha \tan \tfrac{1}{2}\beta.$$

Ex. 5. If a, b, c, d be the sides of a spherical quadrilateral taken in order, δ and δ' the diagonals, and ϕ the arc joining the middle points of the diagonals, shew that

$$\cos \phi = \frac{\cos a + \cos b + \cos c + \cos d}{4 \cos \tfrac{1}{2}\delta \cos \tfrac{1}{2}\delta'}$$

Let the diagonals meet at P and let E and F be their middle points.

Let PC and PD be denoted by x and x' so that $PA=\delta-x$ and $PB=\delta'-x'$. Let the angle APB be θ. Then

$$\cos a=\cos(\delta-x)\cos(\delta'-x')+\sin(\delta-x)\sin(\delta'-x')\cos\theta,$$

$$\cos b=\cos(\delta'-x')\cos x-\sin(\delta'-x')\sin x\cos\theta,$$

$$\cos c=\cos x\cos x'+\sin x\sin x'\cos\theta,$$

and $$\cos d=\cos(\delta-x)\cos x'-\sin(\delta-x)\sin x'\cos\theta.$$

Hence $\cos a+\cos b+\cos c+\cos d$

$$=\{\cos(\delta-x)+\cos x\}\{\cos(\delta'-x')+\cos x'\}$$
$$+\cos\theta\{\sin(\delta-x)-\sin x\}\{\sin(\delta'-x')-\sin x'\}$$
$$=4\cos\tfrac{1}{2}\delta\cos\tfrac{1}{2}\delta'\cos(\tfrac{1}{2}\delta-x)\cos(\tfrac{1}{2}\delta'-x')$$
$$+4\cos\theta\cos\tfrac{1}{2}\delta\cos\tfrac{1}{2}\delta'\sin(\tfrac{1}{2}\delta-x)\sin(\tfrac{1}{2}\delta'-x').$$

Again from the triangle PEF, we have

$$\cos\phi=\cos(\tfrac{1}{2}\delta-x)\cos(\tfrac{1}{2}\delta'-x')+\sin(\tfrac{1}{2}\delta-x)\sin(\tfrac{1}{2}\delta'-x')\cos\theta.$$

Therefore $\cos a+\cos b+\cos c+\cos d=4\cos\phi\cos\tfrac{1}{2}\delta\cos\tfrac{1}{2}\delta',$

or, $$\cos\phi=\frac{\cos a+\cos b+\cos c+\cos d}{4\cos\tfrac{1}{2}\delta\cos\tfrac{1}{2}\delta'}.$$

EXAMPLES

1. In any spherical triangle, shew that

$$\cos a\tan B+\cos b\tan A+\tan C=\cos a\cos b\tan A\tan B\tan C.$$

2. In any spherical triangle, shew that

$$\sin b\sin c+\cos b\cos c\cos A=\sin B\sin C-\cos B\cos C\cos a.$$

(*Cagnoli.*)

(*Dacca Uni.*, 1932.)

3. Prove that

$$2\cos\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b)\tan\tfrac{1}{2}c=\sin b\cos A+\sin a\cos B,$$

and
$$\tan\tfrac{1}{2}(A+a)\tan(B-b)=\tan\tfrac{1}{2}(A-a)\tan\tfrac{1}{2}(B+b).$$

4. If $A=a$, shew that

$$\tan\tfrac{1}{2}a=\frac{\tan\tfrac{1}{2}b\sim\tan\tfrac{1}{2}c}{1-\tan\tfrac{1}{2}b\tan\tfrac{1}{2}c}.$$

(*Science and Art Exam. Papers, 1899; Dacca Uni., 1930.*)

5. Shew that in an equilateral triangle

$$\log\sin\tfrac{1}{2}A+\log\cos\tfrac{1}{2}a+\log 2=0.$$

6. If A and A' denote the angles of an equilateral triangle and its polar reciprocal, shew that

$$\cos A\cos A'=\cos A+\cos A'.$$

(*Science and Art Exam. Papers.*)

7. In any triangle, shew that

$$\cos A=\frac{\cos a\sin b-\sin a\cos b\cos C}{\sin c},$$

and
$$\cos A+\cos B=\frac{2\sin(a+b)\sin^2\tfrac{1}{2}C}{\sin c}.$$

8. Prove that

$$\frac{\cos(B-C)}{\cos(A-C)}=\frac{\tan\tfrac{1}{2}a-\tan\tfrac{1}{2}b\cos C-\tan\tfrac{1}{2}c\cos B}{\tan\tfrac{1}{2}b-\tan\tfrac{1}{2}a\cos C-\tan\tfrac{1}{2}c\cos A}$$

9. Shew that

$$\tan c=\frac{\cos A\cot a+\cos B\cot b}{\cot a\cot b-\cos A\cos B}.$$

10. In a spherical triangle, shew that

$$\sin (S-A)=\frac{1+\cos a-\cos b-\cos c}{4 \cos \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c},$$

and

$$\cos (s-a)=\frac{1-\cos A+\cos B+\cos C}{4 \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C}.$$

11. Shew that

$$\cos^2 \tfrac{1}{2}c=\cos^2 \tfrac{1}{2}(a-b) \cos^2 \tfrac{1}{2}C+\cos^2 \tfrac{1}{2}(a+b) \sin^2 \tfrac{1}{2}C,$$

and $\sin^2 \frac{1}{2}c=\sin^2 \frac{1}{2}(a-b) \cos^2 \frac{1}{2}C+\sin^2 \frac{1}{2}(a+b) \sin^2 \frac{1}{2}C.$

12. Shew that

$$\tan^2 \tfrac{1}{2}c=\frac{\tan^2 \frac{1}{2}a-2 \tan \frac{1}{2}a \tan \frac{1}{2}b \cos C+\tan^2 \frac{1}{2}b}{1+2 \tan \frac{1}{2}a \tan \frac{1}{2}b \cos C+\tan^2 \frac{1}{2}a \tan^2 \frac{1}{2}b}.$$

[Substitute $\dfrac{1-\tan^2 \frac{1}{2}a}{1+\tan^2 \frac{1}{2}a}$ for $\cos a$ and $\dfrac{2 \tan \frac{1}{2}a}{1+\tan^2 \frac{1}{2}a}$ for $\sin a$, etc., in the formula of Art. 3.1.]

13. Shew that

$$\Sigma \tan \tfrac{1}{2}a \frac{\sin \frac{1}{2}(B-C)}{\sin \frac{1}{2}B \sin \frac{1}{2}C}=0.$$

[Substitute values for $\tan \frac{1}{2}a$, etc.]

14. If δ and δ' denote the lengths of the internal and external bisectors of the angle C of a spherical triangle and terminated by the side AB, shew that

$$\cot \delta=\frac{\cot a+\cot b}{2 \cos \frac{1}{2}C},$$

and

$$\cot \delta'=\frac{\cot a \sim \cot b}{2 \sin \frac{1}{2}C}.$$

15. If δ_1, δ_2 and δ_3 denote the bisectors of the internal angle of a spherical triangle, shew that

$$\cot \delta_1 \cos \tfrac{1}{2}A+\cot \delta_2 \cos \tfrac{1}{2}B+\cot \delta_3 \cos \tfrac{1}{2}C=\cot a+\cot b+\cot c.$$

(*Dacca Uni.*, *1930.*)

16. If δ'_1, δ'_2 and δ'_3 denote the bisectors of the external angles of a spherical triangle, shew that

$$\cot \delta'_1 \sin \tfrac{1}{2}A + \cot \delta'_2 \sin \tfrac{1}{2}B + \cot \delta'_3 \sin \tfrac{1}{2}C = 0.$$

17. If s and s' are the segments of the base made by the perpendicular from the vertex C, and σ and σ' those made by the bisector of the vertical angle, shew that

$$\tan \frac{s-s'}{2} \tan \frac{\sigma-\sigma'}{2} = \tan^2 \frac{a-b}{2}.$$

(*Dublin Univ. Exam. Papers.*)

18. If a ship be proceeding uniformly along a great circle and l_1, l_2 and l_3 be the latitudes observed at equal intervals of time, in each of which the distance traversed is s, shew that

$$s = r \cos^{-1} \frac{\sin \frac{1}{2}(l_1 + l_3) \cos \frac{1}{2}(l_1 - l_3)}{\sin l_2},$$

r denoting the radius of the Earth.

19. If ϕ denotes the angle between the bisector of the vertical angle C of a spherical triangle and the perpendicular from C on the base AB, shew that

$$\tan \phi = \frac{\sin (a-b)}{\sin (a+b)} \cot \tfrac{1}{2}C.$$

20. If in any spherical triangle $C = A + B$, shew that

$$1 - \cos a - \cos b + \cos c = 0.$$

21. If in any spherical triangle $a + b = \pi + c$, shew that

$$1 + \cos A + \cos B - \cos C = 0.$$

22. If in a spherical triangle $b + c = \pi$, shew that

$$\sin 2B + \sin 2C = 0.$$

(*Dacca Uni., 1932.*)

23. If A, B, C and D are four points on the surface of a sphere and θ is the angle between the arcs AB and CD, shew that

$\cos AC \cos BD - \cos AD \cos BC = \sin AB \sin CD \cos \theta$. (*Gauss.*)

24. If a, b, c and d be the sides of a spherical quadrilateral taken in order, δ and δ' be the diagonals, and ψ_1 and ψ_2 be the arcs joining the middle points of the opposite sides a and c, b and d, shew that

$$\cos \psi_1 = \frac{\cos b + \cos d + \cos \delta + \cos \delta'}{4 \cos \frac{1}{2}a \cos \frac{1}{2}c},$$

and

$$\cos \psi_2 = \frac{\cos a + \cos c + \cos \delta + \cos \delta'}{4 \cos \frac{1}{2}b \cos \frac{1}{2}d}.$$

25. If one side of a spherical triangle be divided into four equal parts, and θ_1, θ_2, θ_3 and θ_4 be the angles subtended at the opposite corner by the parts taken in order, then

$$\sin (\theta_1 + \theta_2) \sin \theta_2 \sin \theta_4 = \sin (\theta_3 + \theta_4) \sin \theta_1 \sin \theta_3.$$

26. In an isosceles triangle ABC, each of the base angles are double the vertical angle; shew that

$$\cos a \cos \tfrac{1}{2}a = \cos (c + \tfrac{1}{2}a),$$

where a is one of the equal sides of the triangle.

(*London University Exam. Papers.*)

27. If a, b, c and d be the sides of a spherical quadrilateral taken in order, and δ and δ' be the diagonals intersecting at an angle θ, shew that

$$\cos \theta = \frac{\cos a \cos c - \cos b \cos d}{\sin \delta \sin \delta'}.$$

28. If ψ_1 and ψ_2 be the arcs joining the middle points of pairs of opposite sides a and c, b and d of a spherical quadrilateral, and ϕ the arc joining the middle points of the diagonals δ and δ', shew that

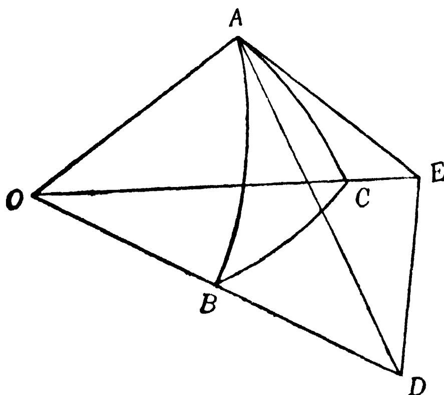
$$\cos \psi_1 \cos \tfrac{1}{2}a \cos \tfrac{1}{2}c + \cos \psi_2 \cos \tfrac{1}{2}b \cos \tfrac{1}{2}d - \cos \phi \cos \tfrac{1}{2}\delta \cos \tfrac{1}{2}\delta' = \tfrac{1}{2}(\cos \delta + \cos \delta').$$

CHAPTER IV

RIGHT-ANGLED TRIANGLES

4.1. Formulae connecting the parts of a right-angled triangle.

Let ABC be a spherical triangle right-angled at C, and let O be the centre of the sphere. At A draw the tangents AD and AE to the arcs AB and AC respectively. They lie in the planes AOB and AOC. Let them meet OB and OC produced at D and E respectively. Join ED.



Since the angle C is a right angle, the planes OCA and OCB are perpendicular to each other.

Also the radius OA is perpendicular to both the tangents AD and AE, and therefore the angles OAD and OAE are right angles and OA is perpendicular to the plane ADE. Also any plane through OA is perpendicular to the plane ADE. Hence the plane OCA is perpendicular to the plane ADE. Thus both the planes ADE and OCB are perpendicular to the plane OCA, and so DE, their line of intersection, is perpendicular to the plane OCA. Therefore the angles OED and AED are right angles.

Now $$\frac{OA}{OD} = \frac{OA}{OE} \cdot \frac{OE}{OD}.$$

that is, $$\cos c = \cos a \cos b. \qquad \ldots \quad (1)$$

Again $$\sin A = \frac{DE}{AD} = \frac{DE}{OD} \cdot \frac{OD}{AD} = \frac{\sin a}{\sin c},$$

that is, $$\sin a = \sin A \sin c. \qquad \ldots \quad (2)$$

Similarly, $$\sin b = \sin B \sin c. \qquad \ldots \quad (3)$$

Also $$\cos A = \frac{AE}{AD} = \frac{AE}{OA} \cdot \frac{OA}{AD} = \frac{\tan b}{\tan c},$$

or, $$\tan b = \cos A \tan c. \qquad \ldots \quad (4)$$

Similarly, $$\tan a = \cos B \tan c. \qquad \ldots \quad (5)$$

And $\tan A = \frac{DE}{AE} = \frac{DE}{OE} \cdot \frac{OE}{AE} = \frac{\tan a}{\sin b}$,

that is, $\tan a = \tan A \sin b$. ... (6)

Similarly, $\tan b = \tan B \sin a$. ... (7)

Multiplying together (6) and (7) we get

$$\tan A \tan B = \frac{\tan a \tan b}{\sin a \sin b} = \frac{1}{\cos a \cos b} = \frac{1}{\cos c},$$

or, $\cos c = \cot A \cot B$. ... (8)

Again dividing (2) by (5) we get

$$\cos a = \frac{\sin A}{\cos B} \cos c = \frac{\sin A}{\cos B} \cos a \cos b,$$

so that $\cos B = \sin A \cos b$. ... (9)

Similarly, from (3) and (4) we have

$\cos A = \sin B \cos a$. ... (10)

The above ten formulae * will enable us to obtain the value of any element of a spherical triangle when two other elements (other than the right angle) are given. All the above formulae could be deduced from those of the previous chapter by putting $C = \frac{1}{2}\pi$.

* These formulae were known to the Hindu Mathematicians and were used by them to solve spherical right-angled triangles. See **A. Arneth**—*Geschichte der reinen Mathematik*. Stuttgart, 1852. **Nasir ed-din al-Tusi** (1201-1274) of Persia collected these formulae into a consistent whole in 1250.

4.2. Some important properties.

Since $\cos c = \cos a \cos b$, it follows that either only one cosine is positive or all of them are positive. Hence in a right-angled triangle either two sides are greater than quadrants and one side less than a quadrant or all the three sides are less than quadrants.

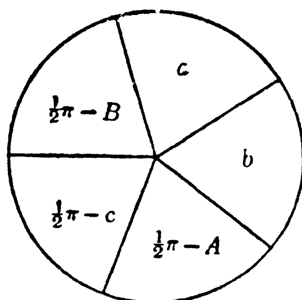
Again since $\tan A = \dfrac{\tan a}{\sin b}$, it follows that $\tan A$ and $\tan a$ are of the same sign. Hence A and a are either both greater than $\frac{1}{2}\pi$ or both less than $\frac{1}{2}\pi$, *i.e., A and a are of the same affection.* Similarly B and b are of the same affection.

4.3. Napier's Rules of Circular parts.*

Napier has given two rules which include in them all the ten formulae established in Art. 4.1. He takes the two sides which include the right angle, the complement of the hypotenuse and the complements of the remaining angles, and calls these the *circular parts* of the triangle. Thus if C be a right angle, the five circular parts are a, b, $\frac{1}{2}\pi - c$, $\frac{1}{2}\pi - A$ and $\frac{1}{2}\pi - B$. He takes a circle and divides it into five sectors and writes one circular part in each sector in the order in which they naturally occur in the triangle.

* These rules are due to **Napier**, and were published by him in his *Mirifici Logarithmorum Canonis Descriptio* in 1614. He calls them theorems, and while he verifies them in the ordinary way, by testing each of the known relations between the parts of a right-angled spherical triangle, he exhibits their true character in relation to the star pentagon with five right angles.

Selecting any one of the five parts, and calling it the *middle part*, the two parts contiguous to it are



called the *adjacent parts* and the remaining two are called the *opposite parts*. Thus if $\frac{1}{2}\pi - c$ is taken as the middle part, then $\frac{1}{2}\pi - A$ and $\frac{1}{2}\pi - B$ will be adjacent parts and a and b the opposite parts.

Napier's Rules are the following:—

(1) sine of the middle part = product of the tangents of the adjacent parts.

(2) sine of the middle part = product of the cosines of the opposite parts.

For example,

$$\sin(\tfrac{1}{2}\pi - c) = \tan(\tfrac{1}{2}\pi - A)\ \tan(\tfrac{1}{2}\pi - B),$$

i.e.,
$$\cos c = \cot A \cot B,$$

which is formula (8) of Art. 4.1.

Again $\sin(\tfrac{1}{2}\pi - c) = \cos a \cos b$

or,
$$\cos c = \cos a \cos b,$$

which is formula (1) of Art. 4.1.

For a proof of the above rules see Napier's *Mirifici Logarithmorum Canonis Descriptio*, 1614, pp. 32-35.

4.4. Quadrantal triangle. When one side of a triangle is a quadrant, it is termed a *Quadrantal triangle*. Evidently it is the polar reciprocal of a right-angled triangle, for if $C=\frac{1}{2}\pi$, we have $c'=\pi-C$ $=\frac{1}{2}\pi$. Hence the formulae for a quadrantal triangle are obtained from those of a right-angled triangle by changing the sides and angles into the supplements of the angles and sides. Thus we have the following formulae when the side c is a quadrant:—

$$\cos C+\cos A \cos B=0. \quad \ldots \quad (1)$$

$$\sin A=\sin a \sin C. \quad \ldots \quad (2)$$

$$\sin B=\sin b \sin C. \quad \ldots \quad (3)$$

$$\tan A+\cos b \tan C=0. \quad \ldots \quad (4)$$

$$\tan B+\cos a \tan C=0. \quad \ldots \quad (5)$$

$$\tan A=\tan a \sin B. \quad \ldots \quad (6)$$

$$\tan B=\tan b \sin A. \quad \ldots \quad (7)$$

$$\cos C+\cot a \cot b=0. \quad \ldots \quad (8)$$

$$\cos b=\sin a \cos B. \quad \ldots \quad (9)$$

$$\cos a=\sin b \cos A. \quad \ldots \quad (10)$$

4.5. Trirectangular triangle. When all the three sides of a spherical triangle are quadrants, it is called a *Trirectangular* or *Triquadrantal Triangle*. Evidently all its angles are also right angles (Ex. 5, p. 26). Thus in a trirectangular triangle the sides and the

angles are all right angles. Each vertex is the pole of the opposite side, and consequently the arc joining a vertex to any point in the opposite side is a quadrant. Since the angle between two radii of the sphere is equal to the arc joining their extremities, it follows that the radii from the centre of the sphere to the vertices of a trirectangular triangle are mutually at right angles. Thus in the figure of Art. 4.7 the radii OA, OB and OC are mutually at right angles.

4.6. Direction Angles and Direction Cosines of a point.

The angles which the radius to a point on the surface of the sphere makes with the radii to the vertices of a trirectangular triangle, at the centre of the sphere, are called the *Direction Angles* of that point, and the cosine of these angles, the *Direction Cosines* of that point. Thus taking ABC to be a trirectangular triangle and P any point on the surface of the sphere whose centre is O, we have the angles POA, POB and POC as the direction angles and $\cos POA$, $\cos POB$ and $\cos POC$ as the direction cosines of the point P. Since the arcs PA, PB and PC measure the angles which OP makes with OA, OB and OC, we can define *direction angles as the angular distances of a point on the surface of a sphere from the vertices of a trirectangular triangle on it.* Thus the arcs PA, PB and PC are the direction angles and

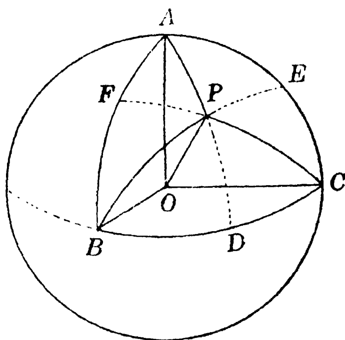
$\cos PA$, $\cos PB$ and $\cos PC$ are the direction cosines of the point P.

Since the angles POA, POB and POC remain the same for all positions of P on the straight line OP, their cosines also remain the same. Thus we get the idea of Direction Cosines of the line OP referred to three rectangular axes OA, OB and OC in solid Geometry.

4.7. Theorem. *If any point P on the surface of a sphere be joined to the vertices of a trirectangular triangle ABC by great circular arcs, then will*

$$\cos^2 PA + \cos^2 PB + \cos^2 PC = 1.$$

i.e., the sum of the squares of the direction cosines of a point on the surface of the sphere is equal to unity.



We have by Art. 3.1

$\cos PA = \cos AB \cos PB + \sin AB \sin PB \cos ABP.$

$= \sin PB \cos ABP$, since AB is a quadrant.

Similarly, $\cos PC = \sin PB \cos PBC = \sin PB \sin ABP$.

Hence, squaring these and adding, we have

$$\cos^2 PA + \cos^2 PC = \sin^2 PB = 1 - \cos^2 PB.$$

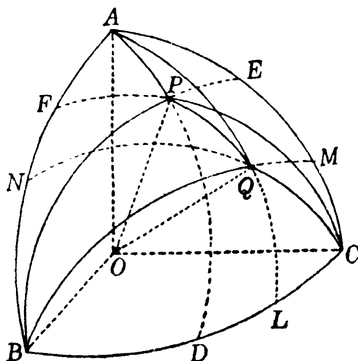
Thus $$\cos^2 PA + \cos^2 PB + \cos^2 PC = 1.$$

Cor. If p_1, p_2 and p_3 be the perpendiculars from the point P on the sides of the triangle ABC, then will

$$\sin^2 p_1 + \sin^2 p_2 + \sin^2 p_3 = 1.$$

4.8. Theorem. *If any two points P and Q on the surface of a sphere be joined to the vertices of a tri-rectangular triangle ABC by great circular arcs, then will*

$$\cos PQ = \cos PA \cos QA + \cos PB \cos QB + \cos PC \cos QC.$$



From the triangle PAQ, we have by Art. 3.1

$$\cos PQ = \cos PA \cos QA + \sin PA \sin QA \cos PAQ.$$

Now $\cos PAQ = \cos(PAC - QAC)$

$$= \cos PAC \cos QAC + \sin PAC \sin QAC$$

$$= \cos PAC \cos QAC + \cos PAB \cos QAB.$$

Therefore

$$\cos PQ = \cos PA \cos QA + \sin PA \sin QA(\cos PAB \cos QAB + \cos PAC \cos QAC).$$

But
$$\cos PB = \sin PA \cos PAB,$$
$$\cos QB = \sin QA \cos QAB,$$
$$\cos PC = \sin PA \cos PAC,$$
$$\cos QC = \sin QA \cos QAC:$$

Hence $\cos PQ = \cos PA \cos QA + \cos PB \cos QB + \cos PC \cos QC.$

This theorem expresses the distance of any two points on the surface of the sphere in terms of their distances from the angular points of a trirectangular triangle.

Cor. If p_1, p_2, p_3; q_1, q_2, q_3 be the perpendiculars from the points P and Q on the sides of the triangle ABC, then will

$$\cos PQ = \sin p_1 \sin q_1 + \sin p_2 \sin q_2 + \sin p_3 \sin q_3.$$

4.9. If we put l, m, n and l', m', n' for the direction cosines of P and Q, and θ for the angular measure of the arc PQ, the two preceding articles give two well-known results of Solid Geometry, *viz.*,

$$(1) \quad l^2+m^2+n^2=1$$

and

$$(2) \quad \cos\theta = ll'+mm'+nn',$$

the direction cosines of OP and OQ being with reference to the three rectangular axes OA, OB and OC.

4.10. Direction Cosines of the Pole of the Arc joining two points on the Surface of the Sphere.

Let P and Q be two points on the surface of the sphere and let H be the pole of the arc PQ. Then by Art. 4.8, we have

$$\cos HP = \cos HA \cos PA + \cos HB \cos PB + \cos HC \cos PC,$$

and

$$\cos HQ = \cos HA \cos QA + \cos HB \cos QB + \cos HC \cos QC.$$

But the arcs HP and HQ are quadrants, hence

$$\cos HA \cos PA + \cos HB \cos PB + \cos HC \cos PC = 0$$

and

$$\cos HA \cos QA + \cos HB \cos QB + \cos HC \cos QC = 0.$$

Solving the above equations, we get

$$\frac{\cos HA}{\cos PB \cos QC-\cos PC \cos QB}$$

$$=\frac{\cos HB}{\cos PC \cos QA-\cos PA \cos QC}$$

$$=\frac{\cos HC}{\cos PA \cos QB-\cos PB \cos QA}$$

$$=\left\{\frac{\cos^2 HA+\cos^2 HB+\cos^2 HC}{\Sigma(\cos PB \cos QC-\cos PC \cos QB)^2}\right\}^{\frac{1}{2}}=\frac{1}{\sin PQ}.^*$$

Thus

$$\cos HA \sin PQ=\cos PB \cos QC-\cos PC \cos QB,$$
$$\cos HB \sin PQ=\cos PC \cos QA-\cos PA \cos QC,$$
$$\cos HC \sin PQ=\cos PA \cos QB-\cos PB \cos QA.$$

Examples worked out

Ex. 1. In a right-angled triangle, if δ be the length of the arc drawn from C perpendicular on the hypotenuse AB meeting it at D, shew that

(1) $\sin^2\delta=\tan AD \tan BD.$

(2) $\tan^2 a=\tan BD \tan c$

and $\tan^2 b=\tan AD \tan c.$

(1) We have from the triangle ACD

$\tan AD=\tan ACD \sin\delta$, by (6) of Art. 4.1.

Similarly, $\tan BD=\tan BCD \sin\delta.$

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* This is obtained from the identical relation

$$(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2$$
$$=(l^2+m^2+n^2)(l'^2+m'^2+n'^2)-(ll'+mm'+nn')^2.$$

Hence multiplying, $\tan AD \tan BD = \sin^2\delta \tan ACD \tan BCD$
$= \sin^2\delta$;

i.e., sine of the perpendicular is the geometric mean between the tangents of the segments of the hypotenuse.

(2) We have from the triangle BCD, by (5) of Art. 4.1

$$\cos B = \frac{\tan a}{\tan c} = \frac{\tan BD}{\tan a}.$$

Hence $\tan^2 a = \tan BD \tan c$.

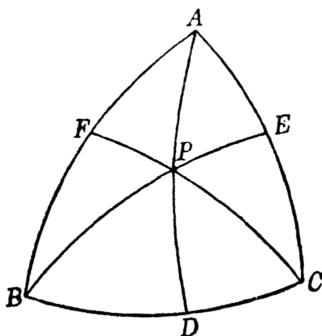
Similarly, $\tan^2 b = \tan AD \tan c$,

i.e., tangent of a side is the geometric mean between tangents of the adjacent segment and the hypotenuse.

Ex. 2. Perpendiculars are drawn from the vertices A, B, C of any triangle, meeting the opposite sides at D, E, F respectively : shew that

$$\tan BD \tan CE \tan AF = \tan DC \tan EA \tan FB.$$

(*Dacca Uni., 1932.*)



Let the perpendiculars meet at the point P. We have from the triangles BPD and CPD, by (6) of Art. 4.1

$$\tan BD = \tan BPD \sin PD$$

and

$$\tan DC = \tan CPD \sin PD.$$

Therefore $$\frac{\tan BD}{\tan DC}=\frac{\tan BPD}{\tan CPD}.$$

Similarly, $$\frac{\tan CE}{\tan EA}=\frac{\tan CPE}{\tan APE},$$

and $$\frac{\tan AF}{\tan FB}=\frac{\tan APF}{\tan BPF}.$$

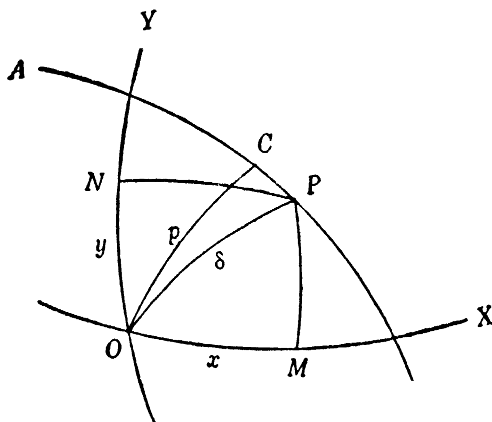
Hence multiplying both sides and noting that

$$B\hat{P}D=A\hat{P}E,\ C\hat{P}D=A\hat{P}F \text{ and } C\hat{P}E=B\hat{P}F,$$

we get $$\frac{\tan BD \tan CE \tan AF}{\tan DC \tan EA \tan FB}=1.$$

Ex. 3. OX and OY are two great circles of a sphere at right angles to each other, P is any point in AB another great circle. $OC(=p)$ is the arc perpendicular to AB from O, making the angle $COX(=a)$ with OX. PM and PN are arcs perpendicular to OX ahd OY respectively : shew that if $OM=x$ and $ON=y$,

$$\cos a \tan x+\sin a \tan y=\tan p.$$



Let OP be denoted by δ and the angle POC by θ.

From the right-angled triangle POM, we have by (4) of Art. 4.1

$$\cos POX = \frac{\tan x}{\tan \delta}.$$

Similarly from the triangle PON, we have

$$\cos POY = \frac{\tan y}{\tan \delta} = \sin POX.$$

Hence

$$\tan x \cos \alpha + \tan y \sin \alpha = \tan \delta \, (\cos \alpha \cos POX + \sin \alpha \sin POX)$$

$$= \tan \delta \cos \theta.$$

But from the triangle POC we have

$$\cos\theta = \frac{\tan p}{\tan \delta}.$$

Hence $\quad \cos \alpha \tan x + \sin \alpha \tan y = \tan p.$

Example

1. If ABC be a triangle in which the angle C is a right angle, prove the following relations—

(1) $2n = \sin a \sin b$.

(2) $2N = \sin a \sin B = \sin A \sin b$.

(3) $\dfrac{n}{N} = \sin c$.

(4) $\sin^2 a \sin^2 b = \sin^2 a + \sin^2 b - \sin^2 c$.

(5) $2 \sin^2 \tfrac{1}{2}c = \sin^2 \tfrac{1}{2}(a+b) + \sin^2 \tfrac{1}{2}(a-b)$.

(6) $\sin a \tan \tfrac{1}{2}A - \sin b \tan \tfrac{1}{2}B = \sin (a-b)$.

(7) $\tan \tfrac{1}{2}B = \dfrac{\sin (s-a)}{\sin s}$.

(8) $\tan^2 \tfrac{1}{2}A = \dfrac{\sin (c-b)}{\sin (c+b)}$.

(9) $\tan \tfrac{1}{2}(A+B) \tan \tfrac{1}{2}(A-B) = \dfrac{\sin (a-b)}{\sin (a+b)}$.

(Dacca Uni., 1930.)

2. In a triangle if C be a right angle and D the middle point of AB, shew that

$$4 \cos^2 \tfrac{1}{2}c \sin^2 CD = \sin^2 a + \sin^2 b.$$

3. If δ be the length of the arc drawn from C perpendicular to the hypotenuse AB, shew that

(1) $\cos^2 \delta = \cos^2 A + \cos^2 B.$

(2) $\cot^2 \delta = \cot^2 a + \cot^2 b.$

(3) $\sin^2 \delta \sin^2 c = \sin^2 a + \sin^2 b - \sin^2 c.$

4. If δ be the length of the arc drawn from C perpendicular to AB in *any* triangle, shew that

$$\cos \delta = \operatorname{cosec} c \,(\cos^2 a + \cos^2 b - 2 \cos a \cos b \cos c)^{\frac{1}{2}}.$$

(*Cal. Uni. M.A. and M.Sc., 1926.*)

5. If the side c of a triangle be a quadrant and δ be the length of the arc drawn at right angles to it from C, shew that—

(1) $\cos^2 \delta = \cos^2 a + \cos^2 b.$

(2) $\cot^2 \delta = \cot^2 A + \cot^2 B.$

(3) $\sin^2 \delta = \cot \theta \cot \phi$, where θ and ϕ are the segments of the angle C.

6. If the side c of a triangle be a quadrant, shew that

(1) $\cos (S-A) \cos (S-B) + \cos (S-C) \cos S = 0.$

(2) $\tan a \tan b + \sec C = 0.$

(3) $2 \cos (S-A) \cos (S-B) = \sin A \sin B.$

7. In the triangle ABC if $C = 90°$, shew that

$$\sin (A+B) = \frac{\cos a + \cos b}{1 + \cos a \cos b}$$

(*Cal. Uni. M.A. and M.Sc., 1931.*)

and

$$\sin (A-B) = \frac{\cos b - \cos a}{1 - \cos a \cos b}.$$

(*Dacca Uni., 1931.*)

8. If $C = 90°$, shew that $\tan S = -\cot \tfrac{1}{2}a \cot \tfrac{1}{2}b.$

9. If one of the sides of a right-angled triangle be equal to the opposite angle, shew that the remaining parts are each equal to 90°.

10. If δ be the length of the bisector of the hypotenuse AB of the right-angled triangle ABC, shew that

$$\sin^2\delta = \frac{\sin^2 a + \sin^2 b}{4\cos^2 \frac{1}{2}c}.$$

11. Shew that the ratio of the cosines of the segments of the base, made by the perpendicular from the vertex, is equal to the ratio of the cosines of the sides.

12. Shew that the ratio of the cosines of the base angles is equal to the ratio of the sines of the segments of the vertical angle made by the perpendicular drawn from it to the base.

13. If α_1, α_2; β_1, β_2 and γ_1, γ_2 be the segments of the sides of a spherical triangle made by the perpendiculars from the opposite vertices, shew that

$$\cos\alpha_1 \cos\beta_1 \cos\gamma_1 = \cos\alpha_2 \cos\beta_2 \cos\gamma_2.$$

14. If p_1, p_2 and p_3, p_4 denote the perpendiculars from the base angles A and B to the internal and external bisectors of the vertical angle C, shew that

$$\sin p_1 \sin p_3 + \sin p_2 \sin p_4 = \sin a \sin b.$$

15. If λ, μ and ν denote the perpendiculars from the vertices of any triquadrantal triangle on a transversal to the sides, shew that

$$\sin^2\lambda + \text{ain}^2\mu + \sin^2\nu = 1.$$

16. ABC is a spherical triangle each of whose sides is a quadrant, and P is any point within the triangle : shew that

$$\cos PA \cos PB \cos PC + \cot BPC \cot CPA \cot APB = 0$$

and

$$\tan ABP \tan BCP \tan CAP = 1.$$

Solution of right-angled triangles.

4.11. We have seen that a triangle has six parts, three sides and three angles, and the formulæ established before shew that if three parts are given, we can determine the remaining three parts, and thus completely solve the triangle. In solving numerical examples, we shall have to make use of logarithmic tables. Six cases present themselves. In these cases the right angle forms a known part and we require to know only two other parts. The angle C is taken to be a right angle in all the following cases.

4.12. Case I. *Having given two sides a and b.*

The remaining elements A, B and c are obtained from the formulæ (6), (7) and (1) of Art. 4.1

$$\cot A = \cot a \sin b,$$
$$\cot B = \cot b \sin a,$$
$$\cos c = \cos a \cos b.$$

The solution is unique and the triangle is always possible.

EXAMPLE

Given $a = 55°18'$, $b = 39°27'$; solve the triangle

To find c, we have

$$\cos c = \cos a \cos b,$$

or, $$10 + L \cos c = L \cos a + L \cos b,$$

or, $$L \cos c = 9{\cdot}6430438.$$

$$c = 63° \; 55' \; 21''.$$

To find A, we have

$$10 + L \cot A = L \cot 55°18' + L \sin 39°27',$$

or, $$L \cot A = 9{\cdot}6434280.$$

$\therefore$ $$A = 66° 15' 6''.$$

To find B, we have

$$10 + L \cot B = L \cot 39°27' + L \sin 55°18',$$

or $$L \cot B = 9{\cdot}9996157.$$

$\therefore$ $$B = 45° 1' 31''.$$

4.13. Case II. *Having given two angles A and B.*

The remaining elements a, b and c are obtained from the formulæ (10), (9) and (8) of Art. 4.1

$$\cos A = \cos a \sin B,$$

$$\cos B = \cos b \sin A,$$

$$\cos c = \cot A \cot B.$$

Here also a, b and c are uniquely determined.

EXAMPLE

Given $A = 64°15'$ and $B = 48°24'$; solve the triangle.

We have

$$L \cos a = 9{\cdot}7641507 \quad \therefore \quad a = 54° 28' 53'',$$

$$L \cos b = 9{\cdot}8675405 \quad \therefore \quad b = 42° 30' 47'',$$

and $$L \cos c = 9{\cdot}6316912 \quad \therefore \quad c = 64° 38' 38''.$$

4.14. Case III. *Having given the hypotenuse c and one side a.*

We have from (2), (5) and (1) of Art 4.1

$$\sin A = \frac{\sin a}{\sin c},$$

$$\cos B = \frac{\tan a}{\tan c},$$

$$\cos b = \frac{\cos c}{\cos a}.$$

The elements B and b are determined without ambiguity, but sin A admits of two values between 0 and π. But since a and A are of the same affection, *i.e.*, they are either both acute or both obtuse, we take that value of A which is of the same affection with a. Thus A is also uniquely determined. The triangle is thus possible.

If a and c are both quadrants, then A is a right angle, but b and B are indeterminate.

4.15. Case IV. *Having given the hypotenuse c and an angle A.*

We have from (2), (4)and (8) of Art. 4.1

$$\sin a = \sin A \sin c,$$

$$\tan b = \cos A \tan c,$$

$$\cot B = \tan A \cos c.$$

Thus B and b are uniquely determined, and as a and A are of the same affection, a is also uniquely determined. Thus the triangle is possible.

If A and c are both right angles, then a is a right angle, but b and B are indeterminate.

4.16. Case V. *Having given one side b and the adjacent angle A.*

The formulæ for determining a, B and c are (4), (6) and (9) of Art. 4.1

$$\tan c = \frac{\tan b}{\cos A},$$

$$\tan a = \tan A \sin b,$$

$$\cos B = \cos b \sin A.$$

Thus a, B and c are uniquely determined.

4.17. Case VI. *Having given one side a and the opposite angle A.*

Here we have from (2), (6) and (10) of Art. 4.1

$$\sin c = \frac{\sin a}{\sin A},$$

$$\sin b = \tan a \cot A,$$

$$\sin B = \frac{\cos A}{\cos a}.$$

Here c, b and B are to be determined from their sines, and between 0 and π there are in general two angles having a given sine. Thus we get two values for each sine, and we expect six different triangles

with the given data. But this is not the case. We must have a and A of the same affection, and since $\sin c$ must be less than unity for c lies between 0 and π, $\sin a$ must be less than $\sin A$, and so a must be less than A when they are both acute or greater than A when they are both obtuse. Otherwise the solution will be impossible. When this condition is satisfied, we get two values for c, and since $\cos c = \cos a \cos b$, we get one value for b for each value of c, and one value for B, because b and B are of the same affection, which is otherwise evident from the relation $\cos c = \cot A \cot B$.

Thus we see that there will be in general two triangles with the given parts. We say in general because if a and A are equal but not right angles, we have b, B and c all right angles and thus we get only one triangle. In this case A is the pole of BC. When a and A are right angles the solution becomes indeterminate.

That we should have two triangles is apparent from the fact that the triangle ABC and its colunar triangle $A''BC$ satisfy the given data, for $A = A''$ and BC is common. If $A = a$, we get one triangle, for the triangle $A''BC$ is symmetrically equal to the triangle ABC.

EXAMPLE

Given $a=51°20'$, $A=62°12'$ and $C=90°$; solve the triangle.

To find c, we have $L \sin c = 10 + L \sin a - L \sin A$

$$= 10 + 9{\cdot}8925 - 9{\cdot}9467 = 9{\cdot}9458.$$

Hence $c = 61° 58'$ or $118° 2'$.

To find b, we have $L \sin b = L \tan a + L \cot A - 10$

$$= 10{\cdot}0968 + 9{\cdot}7220 - 10 = 9{\cdot}8188.$$

Hence $b = 41° 13'$ or $138° 47'$.

To find B, we have $L \sin B = 10 + L \cos A - L \cos a$

$$= 10 + 9{\cdot}6687 - 9{\cdot}7957 = 9{\cdot}8730.$$

Hence $B = 48°17'$ or $131°43'$.

4.18. Application of Napier's analogies in the solution of right-angled triangles.

Napier's analogies can profitably be used in solving right-angled triangles in the three following cases.

Firstly, when the sides a and b are given;

Secondly, when the angles A and B are given;

and Thirdly, when a and B, or b and A are given.

EXAMPLE

Solve the triangle having given

$$a = 64° 30', \; b = 48° 12' \text{ and } C = 90°.$$

To find c, we have $\cos c = \cos a \cos b$,

or, $L \cos c = L \cos 64° 30' + L \cos 48° 12' - 10$

$$= 9{\cdot}6340 + 9{\cdot}8238 - 10 = 9{\cdot}4578.$$

Hence $c = 73° 19' 28''$.

To find A and B, we have from Napier's first analogy

$$\tan \tfrac{1}{2}(A+B)=\frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)},$$

or, $L \tan \frac{1}{2}(A+B)=10+L \cos 8^\circ\ 9'-L \cos 56^\circ\ 21'$

$=10+9{\cdot}9956-9{\cdot}7436$

$=10{\cdot}2520.$

Hence $\frac{1}{2}(A+B)=60^\circ\ 45'\ 40''.$

Similarly $L \tan \frac{1}{2}(A-B)=10+L \sin 8^\circ\ 9'-L \sin 56^\circ\ 21'$

$=10+9{\cdot}1516-9{\cdot}9204$

$=9{\cdot}2312.$

Hence $\frac{1}{2}(A-B)=9^\circ\ 39'\ 53''.$

$\therefore$ $A=70^\circ\ 25'\ 33''$ and $B=51^\circ\ 5'\ 47''.$

4.19. Solution of oblique-angled triangles.

As in the case of right-angled triangles, six different cases present here also, and when we are given any three of the parts, we can determine the remaining three parts by making use of some of the formulæ of Chapter III. We shall not go in details but finish this chapter by giving an application of Napier's analogies to solve an oblique-angled triangle.

EXAMPLE

Solve the triangle having given

$A=130^\circ\ 5'\ 22{\cdot}41''$, $B=32^\circ\ 26'\ 6{\cdot}41''$ and $c=51^\circ\ 6'\ 11{\cdot}6''.$

From Napier's third analogy, we have

$L \tan \frac{1}{2}(a+b)=L \cos \frac{1}{2}(A-B)-L \cos \frac{1}{2}(A+B)+L \tan \frac{1}{2}c$

$=9{\cdot}81844-9{\cdot}18158+9{\cdot}67950$

$=10{\cdot}31636.$

Hence $\frac{1}{2}(a+b)=64^\circ\ 14'\ 7''.$

Similarly $L \tan \frac{1}{2}(a-b) = L \sin \frac{1}{2}(A-B) - L \sin \frac{1}{2}(A+B) + L \tan \frac{1}{2}c$

$$= 9{\cdot}87663 - 9{\cdot}99493 + 9{\cdot}67950,$$

whence $\frac{1}{2}(a-b) = 20^\circ 0' 22''$.

$\therefore$ $a = 84^\circ\ 14'\ 29''$ and $b = 44^\circ\ 13'\ 45''$.

To find C, we use Delambre's third analogy, whence

$L \sin \frac{1}{2}C = L \cos \frac{1}{2}(A+B) - L \cos \frac{1}{2}(a+b) + L \cos \frac{1}{2}c$

$$= 9{\cdot}18158 - 9{\cdot}63816 + 9{\cdot}95529.$$

Hence $\frac{1}{2}C = 18^\circ 22' 43''$,

or, $C = 36^\circ 45' 26''$.

The value of C can also be obtained from Napier's first analogy.

EXAMPLES.

Solve the following triangles having given

1.	$a=37^\circ\ 48'\ 12''$,	$b=59^\circ\ 44'\ 16''$,	$C=90^\circ$.
Ans.	$A=41^\circ\ 55'\ 45''$,	$B=70^\circ\ 19'\ 15''$,	$c=66^\circ\ 32'\ 6''$.
2.	$a=54^\circ\ 16'$,	$b=33^\circ\ 12'$,	$C=90^\circ$.
Ans.	$A=68^\circ\ 29'\ 53''$.	$B=38^\circ\ 52'\ 26''$,	$c=60^\circ\ 44'\ 46''$.
3.	$A=36^\circ$,	$B=60^\circ$,	$C=90^\circ$.
Ans.	$a=20^\circ\ 54'\ 18{\cdot}5''$,	$b=31^\circ\ 43'\ 3''$,	$c=37^\circ\ 21'\ 38{\cdot}5''$.
4.	$a=59^\circ\ 28'\ 27''$,	$A=66^\circ\ 7'\ 20''$,	$C=90^\circ$.
Ans.	$b=48^\circ\ 39'\ 16''$,	$B=52^\circ\ 50'\ 20''$,	$c=70^\circ\ 23'\ 42''$.
or	$b=131^\circ\ 20'\ 44''$,	$B=127^\circ\ 9'\ 40''$,	$c=109^\circ\ 36'\ 18''$.
5.	$A=23^\circ\ 27'$,	$B=7^\circ\ 15'$	$c=74^\circ\ 29'$.
Ans.	$a=60^\circ$,	$b=15^\circ\ 56'$,	$C=153^\circ\ 44'$.
6.	$a=138^\circ\ 4'$,	$b=109^\circ\ 41'$,	$c=90^\circ$.
Ans.	$A=142^\circ\ 11'\ 38''$,	$B=120^\circ\ 15'\ 57''$,	$C=113^\circ\ 28'\ 2''$.
7.	$A=46^\circ\ 45'$,	$c=75^\circ\ 40'$,	$C=90^\circ$.
Ans.	$a=44^\circ\ 53'\ 9{\cdot}4$,	$b=69^\circ\ 32'\ 55''$,	$B=75^\circ\ 15'\ 22''$.

CHAPTER V

PROPERTIES OF SPHERICAL TRIANGLES

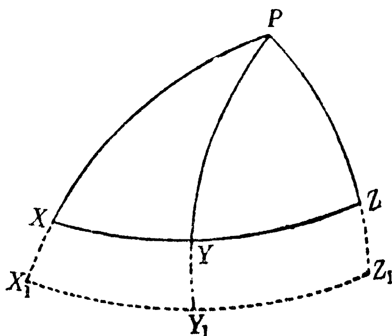
5.1. Relations between the arcs joining three doints on a great circle and any other point.

Theorem. *If X, Y, Z be three points on a great circle, and P any other point, on the sphere, then will*

$$\cos PX \sin YZ + \cos PY \sin ZX + \cos PZ \sin XY = 0 \quad \ldots(1)$$

and

$$\cot PX \sin YPZ + \cot PY \sin ZPX + \cot PZ \sin XPY = 0 \quad \ldots(2)$$



From the triangles PXY and PYZ we have

$$\cos PX = \cos PY \cos XY + \sin PY \sin XY \cos PYX$$

and $\cos PZ = \cos PY \cos YZ + \sin PY \sin YZ \cos PYZ.$

But $\cos PYX = -\cos PYZ$; hence

$$\cos PZ = \cos PY \cos YZ - \sin PY \sin YZ \cos PYX.$$

Eliminating $\cos PYX$ from these two equations, we have

$$(\cos PX - \cos PY \cos XY) \sin YZ$$
$$+ (\cos PZ - \cos PY \cos YZ) \sin XY = 0,$$

or, $\cos PX \sin YZ - \cos PY \sin XZ + \cos PZ \sin XY = 0.$

Writing $XZ = -ZX$, by measuring arcs in one direction as positive and, in the opposite direction, as negative, we have

$$\cos PX \sin YZ + \cos PY \sin ZX + \cos PZ \sin XY = 0 \quad \ldots(1)$$

Again by Art. 3.15 we have from the triangles PXY and PYZ

$$\sin PY \cot PX = \cos PY \cos XPY + \sin XPY \cot PYX$$

and

$$\sin PY \cot PZ = \cos PY \cos YPZ + \sin YPZ \cot PYZ.$$

Multiplying these two equations by $\sin YPZ$ and $\sin XPY$ respectively and adding, we have

$$\sin PY (\cot PX \sin YPZ + \cot PZ \sin XPY)$$
$$= \cos PY \sin XPZ,$$

or, putting $XPZ = -ZPX$, we have

$$\cot PX \sin YPZ + \cot PY \sin ZPX$$
$$+ \cot PZ \sin XPY = 0 \ldots \quad (2)$$

5.2. Particular cases.

(1) **Median.** If Y be the middle point of XZ, then PY is the median of the triangle PXZ, and (1) gives

$$\cos PY = \frac{(\cos PX + \cos PZ)\sin XY}{\sin XZ},$$

or, $$\cos PY = \frac{\cos PX + \cos PZ}{\cos XY + \cos YZ}.$$

Thus if m be the length of the median bisecting the side a of the triangle ABC, we have

$$\cos m = \frac{\cos b + \cos c}{2 \cos \frac{1}{2}a}.*$$

(2) **Internal Bisector of an angle.** If PY bisects the angle P, we have from (2)

$$\cot PY = \frac{\sin XPY\,(\cot PX + \cot PZ)}{\sin XPZ}$$

$$= \frac{\cot PX + \cot PZ}{\cos XPY + \cos YPZ}.$$

Thus the internal bisector δ of the angle A of the triangle ABC is given by

$$\cot \delta = \frac{1}{2 \cos \frac{1}{2}A}(\cot b + \cot c).$$

* **Gudermann,** *Niedere Sphärik,* § 400.

(3) **External Bisector of an angle.** If PZ bisects externally the angle XPY, then

$$Y\hat{P}Z = \tfrac{1}{2}\pi - \tfrac{1}{2}\,X\hat{P}Y \text{ and } X\hat{P}Z = \tfrac{1}{2}\pi + \tfrac{1}{2}\,X\hat{P}Y,$$

so that

$$\cot PZ = \frac{\cot PY - \cot PX}{2 \sin \frac{1}{2}\,XPY}.$$

Thus the external bisector δ' of the angle A of the triangle ABC is given by

$$\cot \delta' = \frac{1}{2 \sin \frac{1}{2}A}(\cot b - \cot c).$$

(4) If XZ be a quadrant, we have

$$\cos PY = \cos PX \sin YZ + \cos PZ \sin XY.$$

Thus if the base BC be a quadrant, and a point D be taken in it, we have

$$\cos AD = \cos c \sin DC + \cos b \sin BD.$$

5.3. Spherical Perpendiculars. Let the arcs PX, PY and PZ when produced meet another great circle at right angles at the points X_1, Y_1 and Z_1 respectively, then P is the pole of the great circle $X_1Y_1Z_1$, and each of the arcs PX_1, PY_1 and PZ_1 is a quadrant. (See fig. of Art 5.1.) Hence

$$\cos PX = \sin XX_1, \quad \cos PY = \sin YY_1$$

and $\quad \cos PZ = \sin ZZ_1,$

and (1) of Art. 5.1 becomes

$$\sin XX_1 \sin YZ + \sin YY_1 \sin ZX + \sin ZZ_1 \sin XY = 0 \quad (3)$$

Similarly (2) of Art. 5.1 gives

$$\tan XX_1 \sin YPZ + \tan YY_1 \sin ZPX + \tan ZZ_1 \sin XPY = 0 \quad \ldots \quad (4)$$

Since the angle between any two arcs PX and PY is measured by the intercept made by them on the great circle $X_1Y_1Z_1$, *i.e.*, by X_1Y_1 (Art. 1.8), we get

$$\tan XX_1 \sin Y_1Z_1 + \tan YY_1 \sin Z_1X_1 + \tan ZZ_1 \sin X_1Y_1 = 0 \quad \ldots \quad (5)$$

These are the relations connecting the spherical perpendiculars XX_1, YY_1 and ZZ_1 from the points X, Y and Z on the great circle $X_1Y_1Z_1$.

5.4. Theorem. *If three arcs meet at a point, the ratio of the sines of the arcs drawn from any point on one of the arcs, perpendicular to the other two, is constant.*

Let OA, OB and OC be the three arcs and let α and β be the lengths of the perpendiculars from a point P in OB on the arcs OA and OC respectively.

Then from the two right-angled triangles, we have

$$\sin OP = \frac{\sin \alpha}{\sin AOP} = \frac{\sin \beta}{\sin COP},$$

or, $$\frac{\sin \alpha}{\sin \beta} = \frac{\sin AOP}{\sin COP}, \text{ which is constant,}$$

for it is independent of the position of P on the arc OB.

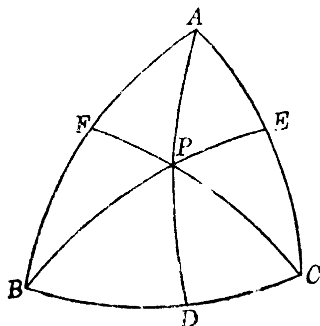
Conversely, if from any point P' in OB, perpendiculars α' and β' are drawn on OA and OC so as to satisfy the relation

$$\frac{\sin \alpha'}{\sin \beta'} = \frac{\sin \alpha}{\sin \beta},$$

then P' will lie on the great circle through O and P, namely OB.

5.5. Concurrency of three arcs.

Theorem. *If three arcs joining a given point with the angular points of a triangle meet the opposite sides, the product of the sines of the alternate segments of the sides are equal.*



Let the arcs joining A, B and C with the given point P meet the opposite sides in D, E and F respectively.

Then from the triangles APF and BPF we have

$$\frac{\sin AF}{\sin AP}=\frac{\sin APF}{\sin AFP} \quad \text{and} \quad \frac{\sin FB}{\sin BP}=\frac{\sin BPF}{\sin BFP},$$

so that $$\frac{\sin AF}{\sin FB}=\frac{\sin AP}{\sin BP}\cdot\frac{\sin APF}{\sin BPF}.$$

Similarly $$\frac{\sin BD}{\sin DC}=\frac{\sin BP}{\sin CP}\cdot\frac{\sin BPD}{\sin CPD},$$

and $$\frac{\sin CE}{\sin EA}=\frac{\sin CP}{\sin AP}\cdot\frac{\sin CPE}{\sin APE}.$$

Hence multiplying the corresponding sides of the three equalities and noting that

$$B\hat{P}D=A\hat{P}E, \quad C\hat{P}D=A\hat{P}F \text{ and } C\hat{P}E=B\hat{P}F,$$

we have $$\frac{\sin AF}{\sin FB}\cdot\frac{\sin BD}{\sin DC}\cdot\frac{\sin CE}{\sin EA}=1.$$

The corresponding theorem for a plane triangle is Ceva's theorem.*

5.6. The converse theorem can also be easily proved. Several theorems on concurrency of arcs are immediately deducible from it. Thus

The perpendiculars drawn from the vertices of a spherical triangle to the opposite sides meet at a point.†

* See **Russel's** *Pure Geometry*, Chap. I.

† **Gudermann**, *Niedere Sphärik*, §68; **Schulz**, *Sphärik* II, §47.

The bisectors of the angles of a spherical triangle meet at a point.*

The arcs joining the angular points of a spherical triangle with the middle points of the opposite sides meet at a point.

5.7. Theorem. *If three arcs passing through the vertices of a triangle be concurrent, the products of the sines of the alternate segments of the angles of the triangle are equal.*

Let the arcs AD, BF and CF meet at P and divide the angles A, B, C of the triangle ABC into the segments A_1, A_2 ; B_1, B_2 and C_1, C_2. (See fig. of Art. 5.5.)

Then from the triangles ABD and ACD, we have

$$\frac{\sin BD}{\sin A_1}=\frac{\sin c}{\sin ADB} \quad \text{and} \quad \frac{\sin DC}{\sin A_2}=\frac{\sin b}{\sin ADC}.$$

Hence $$\frac{\sin BD}{\sin DC}=\frac{\sin A_1}{\sin A_2}\cdot\frac{\sin c}{\sin b}.$$

Similarly $$\frac{\sin CE}{\sin EA}=\frac{\sin B_1}{\sin B_2}\cdot\frac{\sin a}{\sin c},$$

and $$\frac{\sin AF}{\sin FB}=\frac{\sin C_1}{\sin C_2}\cdot\frac{\sin b}{\sin a}.$$

* First proved by **Menelaus.**

Therefore $\frac{\sin A_1}{\sin A_2} \cdot \frac{\sin B_1}{\sin B_2} \cdot \frac{\sin C_1}{\sin C_2} = 1.$

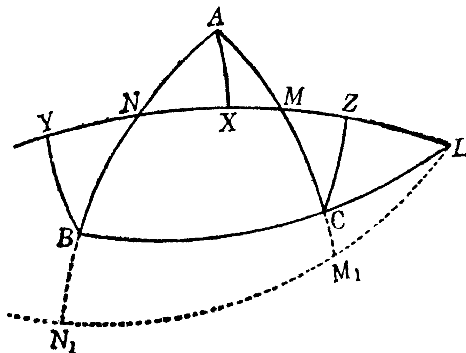
This is also another criterion for the concurrency of three arcs.

The converse case can also be easily proved.

5.8. Concyclic points. Spherical transversal.

Theorem. *If a great circle intersects the sides of a triangle ABC at the points L, M and N, then will*

$$\frac{\sin AN}{\sin NB} \cdot \frac{\sin BL}{\sin LC} \cdot \frac{\sin CM}{\sin MA} = -1.$$



Draw AX, BY and CZ perpendiculars on the great circle LMN. Then from the triangle ANX we have

$$\sin AX = \sin AN \sin ANX.$$

Similarly from the triangle BNY, we have

$$\sin BY = \sin NB \sin B\hat{N}Y.$$

Hence $$\frac{\sin AN}{\sin NB} = \frac{\sin AX}{\sin BY}.$$

Similarly $$\frac{\sin BL}{\sin CL} = \frac{\sin BY}{\sin CZ} \text{ and } \frac{\sin CM}{\sin MA} = \frac{\sin CZ}{\sin AX}.$$

Hence multiplying and writing $-\sin LC$ for $\sin CL$, we have

$$\frac{\sin AN}{\sin NB} \frac{\sin BL}{\sin LC} \frac{\sin CM}{\sin MA} = -1.$$

This theorem along with its analogue for plane triangle was obtained by Menelaus.*

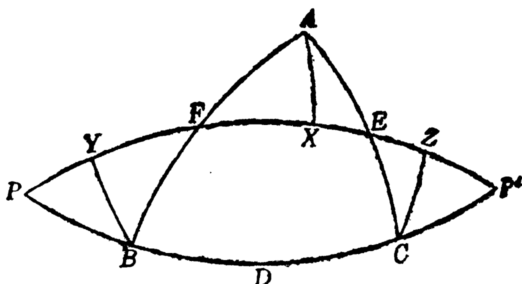
Its converse is also true, namely if three points L, M, N be taken on the sides of a triangle satisfying the above relation, then they will lie on a great circle.

Note 1.—Any transversal must cut either one or all the three sides of the triangle externally. Thus the arc LMN cuts only the side BC externally whereas the arc LM_1N_1 cuts all the sides externally. Hence there will always be the negative sign.

Note 2.—Several formulae for right-angled triangles are easily deducible from Menelaus' theorem. Thus if $C = 90°$ and AN and AM are quadrants, then L will be the pole of AC and the theorem becomes $\cos c = \cos a \cos b$. Again the triangle NBL with AC as transversal gives $\sin a = \sin A \sin c$. Other formulae are similarly obtained by taking any three arcs as forming a triangle with the forth one as the transversal.

* In Greek geometry this theorem is known by the name of *Regula Sex Quantitatum*. See *Sphaerica* by **Menelaus** or *Des Claudius Ptolemaus Handbuch der Astronomie* by **Karl Manitius**, Bd. I, pp. 45-51.

5.9. Theorem. *The great circle bisecting the sides of a triangle intersects the base in points which are equidistant from the middle point of the base.*



Let ABC be the triangle and let D, E and F be the middle points of the sides BC, CA and AB respectively. Draw the secondaries AX, BY and CZ on EF. Let EF and BC when produced meet at the points P and P'. Clearly these are two diametrically opposite points.

Now in the triangles AFX and BFY, we have

$$AF = FB, \quad A\hat{X}F = B\hat{Y}F \quad \text{and} \quad A\hat{F}X = B\hat{F}Y.$$

Hence the triangles are equal in all respects so

that $\qquad AX = BY.$

Similarly $\qquad AX = CZ.$

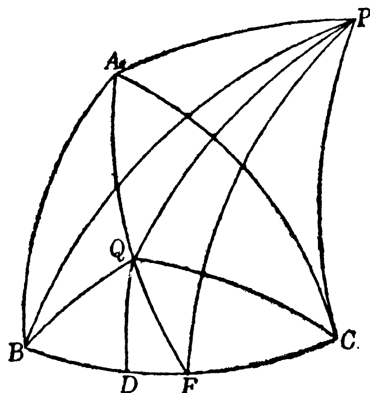
Therefore $\qquad AX = BY = CZ.$

Again from the equality of the triangles BPY and $CP'Z$, we easily get $BP = CP'$, and as $BD = CD$,

we have $\qquad DP = DP' =$ a quadrant.

5.10. Casey's Theorem.* *If two points P and Q be taken on the surface of a sphere, of which Q is within a spherical triangle ABC, and if* $2n_1$, $2n_2$ *and* $2n_3$ *be the sines of the triangles QBC, QCA and QAB, then*

$$n_1 \cos PA + n_2 \cos PB + n_3 \cos PC = n \cos PQ.$$



Join P and Q to the points A, B, C.

Produce AQ to meet BC in F. Join PF and PQ.

Then since B, F and C lie on a great circle, and P is any other point, we have by Art. 5.1

$$\cos PB \sin FC + \cos PC \sin BF = \cos PF \sin BC.$$

Similarly for the points A, Q, F and P, we have

$$\cos PA \sin QF + \cos PF \sin AQ = \cos PQ \sin AF.$$

* Dr. Casey, *Spherical Trigonometry*, p. 81.

Hence eliminating $\cos PF$ from these two equations, we get

$$\cos PA \sin QF \sin BC + \cos PB \sin FC \sin AQ + \cos PC \sin BF \sin AQ = \cos PQ \sin AF \sin BC.$$

If QD be drawn at right angles to BC, then

$$\sin QF = \frac{\sin QD}{\sin F},$$

so that

$$\sin QF \sin BC = \frac{\sin QD \sin BC}{\sin F} = \frac{2n_1}{\sin F}, \text{ by Ex. 4, p. 40.}$$

Similarly

$$\sin AQ \sin FC = \frac{2n_2}{\sin F}, \quad \sin AQ \sin BF = \frac{2n_3}{\sin F},$$

and $$\sin AF \sin BC = \frac{2n}{\sin F}.$$

Hence we have

$$n_1 \cos PA + n_2 \cos PB + n_3 \cos PC = n \cos PQ.$$

5.11. Normal co-ordinates of a point. If from a point P perpendiculars α, β, γ are drawn to the sides of a triangle ABC, then $\sin \alpha$, $\sin \beta$ and $\sin \gamma$ are called the *Normal co-ordinates* of P with respect to the triangle.

Normal co-ordinates are clearly analogous to *trilinear co-ordinates* with respect to a plane triangle.

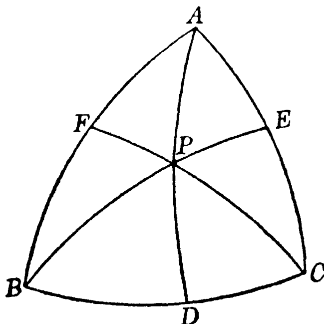
If $2n_1$, $2n_2$ and $2n_3$ be the sines of the triangles *PBC*, *PCA* and *PAB*, we have

$\sin \alpha \sin a = 2n_1$, $\sin \beta \sin b = 2n_2$ and $\sin \gamma \sin c = 2n_3$.

When the ratios of the co-ordinates are known, the point is determined.

EXAMPLE

Find the normal co-ordinates of the point where the perpendiculars from the angular points to the opposite sides meet.



Let the perpendiculars *AD*, *BE* and *CF* meet at *P*.

Now from the triangles *ABD* and *ACD*, we have by (9) of Art. 4.1

$$\cos B = \cos AD \sin BAD, \text{ and } \cos C = \cos AD \sin CAD.$$

Hence $\dfrac{\cos B}{\cos C} = \dfrac{\sin BAD}{\sin CAD} = \dfrac{\sin \gamma}{\sin \beta}$, by Art. 5.4.

Similarly, $\dfrac{\cos C}{\cos A} = \dfrac{\sin \alpha}{\sin \gamma}$.

Hence $\sin \alpha \cos A = \sin \beta \cos B = \sin \gamma \cos C$

i.e., $\sin \alpha$, $\sin \beta$ and $\sin \gamma$ are respectively proportional to $\cos B \cos C$, $\cos C \cos A$ and $\cos A \cos B$.

5.12. Normal co-ordinates with respect to a trirectangular triangle. Their fundamental properties.

We have seen (Art. 4.5) that the arc joining the vertex of a trirectangular triangle to any point in the opposite side is a quadrant. Hence if P be any point on the sphere, and D, E, F the points where AP, BP and CP meet the opposite sides, then PD, PE and PF will be complementary to AP, BP and CP respectively. (See figure of Art. 4.7.)

Now the Normal co-ordinates of P are sin PD, sin PE and sin PF. Hence with respect to the trirectangular triangle they are cos AP, cos BP and cos CP, and these are generally represented by l, m and n. In fact l, m, n are the direction cosines of OP referred to three rectangular axes OA, OB and OC, O being the centre of the sphere.

They satisfy the following properties—

$$(i) \qquad l^2+m^2+n^2=1$$

and $$(ii) \qquad ll'+mm'+nn'=\cos PQ,$$

l, m, n and l', m', n' being the normal co-ordinates of two points P and Q on the sphere.

EXAMPLES

1. If D be any point in the side BC of the triangle ABC, shew that $\cot AD \sin BAC = \cot AC \sin BAD + \cot AB \sin DAC$.

2. If two sides of a spherical triangle be supplementary, prove that the median passing through their intersection is a quadrant.

(*R.U.I.*, *1895.*)

3. The medians of a triangle ABC intersect at P and meet the opposite sides at D, E, F respectively : shew that

(i) $\sin PA : \sin PD :: 2 \cos \frac{1}{2}a : 1,$

(ii) $\sin PB : \sin PE :: 2 \cos \frac{1}{2}b : 1,$

(iii) $\sin PC : \sin PF :: 2 \cos \frac{1}{2}c : 1.$

4. From any three points on a great circle, secondaries x, y, z; x', y', z' and x'', y'', z'' are drawn to the sides of a triangle : shew that

$$\begin{vmatrix} \sin x, & \sin y, & \sin z \\ \sin x', & \sin y', & \sin z' \\ \sin x'', & \sin y'', & \sin z'' \end{vmatrix} = 0.$$

5. Three points P, Q and R lie on a great circle, and X, Y and Z are three other points on the sphere : shew that

$$\begin{vmatrix} \cos PX, & \cos PY, & \cos PZ \\ \cos QX, & \cos QY, & \cos QZ \\ \cos RX, & \cos RY, & \cos RZ \end{vmatrix} = 0.$$

6. If the bisectors of the angles of the triangle ABC meet at P, shew that

(i) $$\frac{\sin BPC}{\sin AP} : \frac{\sin CPA}{\sin BP} : \frac{\sin APB}{\sin CP} = \sin a : \sin b : \sin c.$$

(ii) $\sin^2 AP : \sin^2 BP : \sin^2 CP$

$$= \frac{\sin (s-a)}{\sin a} : \frac{\sin (s-b)}{\sin b} : \frac{\sin (s-c)}{\sin c}$$

7. Find the Normal co-ordinates of the point where the arcs joining the angular points of a triangle to the middle points of the opposite sides meet.

Ans. Proportional to sin B sin C, sin C sin A and sin A sin B.

8. If the internal bisectors of the angles of the triangle ABC intersect at P and meet the opposite sides in D, E and F respectively, shew that

$$\frac{\sin PD}{\sin a \sin AD}=\frac{\sin PE}{\sin b \sin BE}=\frac{\sin PF}{\sin c \sin CF}$$

$$=\frac{1}{\{\sin^2 s+\sin s \sin(s-a)\sin(s-b)\sin(s-c)\}^{\frac{1}{2}}}.$$

(*R.U.I.*, *1896.*)

9. If α, α'; β, β' and γ, γ' be the segments of the perpendiculars to the sides of a spherical triangle drawn from the opposite vertices, shew that

$$\tan\alpha\tan\alpha'=\tan\beta\tan\beta'=\tan\gamma\tan\gamma'$$

and

$$\frac{\cos(\alpha+\alpha')}{\cos\alpha\cos\alpha'}=\frac{\cos(\beta+\beta')}{\cos\beta\cos\beta'}=\frac{\cos(\gamma+\gamma')}{\cos\gamma\cos\gamma'}.$$

10. ABC is a spherical triangle, E is the middle point of BC, and AD is drawn at right angles to BC; shew that

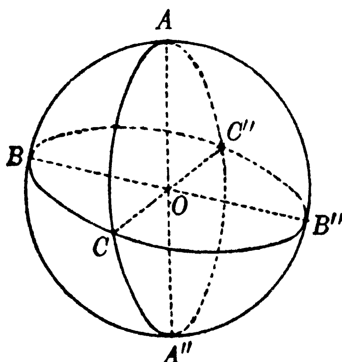
$$\tan ED\sin(B+C)=\tan\tfrac{1}{2}a\sin(B-C).$$

(*Sci. and Art.*, *1894.*)

CHAPTER VI

AREA OF SPHERICAL TRIANGLE. SPHERICAL EXCESS

6.1. Area of a spherical triangle. Girard's theorem.*



Let ABC be a spherical triangle. Produce the sides AB and AC. They will meet at A'' where A'' is the point diametrically opposite to A (Art. 2.11). Thus we get a lune $ABA''CA$ with the angle A. Similarly BC and BA produced give the lune

* This theorem is due to **Girard** and was published by him in 1629 in his *Invention nouvelle en Algèbre.* A rigorous proof of it was given by **Cavalieri** in his *Directorium generale uranio-metricum* in 1632.

$BCB''AB$ of the angle B, and CA and CB produced give the lune $CAC''BC$ of the angle C. The triangle ABC forms a part of each of these three lunes. Let r be the radius of the sphere.

Then $\quad ABC + A''BC = \text{lune } ABA''CA = 2Ar^2,$

$\quad ABC + AB''C = \text{lune } BCB''AB = 2Br^2,$

and $\quad ABC + ABC'' = \text{lune } CAC''BC = 2Cr^2,$

by Art. 2.11.

Now the triangle ABC'' is antipodal to $A''B''C$ and hence they are equal in area (Art. 2.12). Hence putting $A''B''C$ in place of ABC'' and adding the three equalities above, we get

$$2 \text{ triangle } ABC + \text{area of hemisphere} = 2(A + B + C)r^2,$$

or, $$\text{triangle } ABC + \pi r^2 = (A + B + C)r^2.$$

Therefore

$$\text{area of the triangle } ABC = (A + B + C - \pi)r^2 \quad \ldots \quad (1)$$

The expression $A + B + C - \pi$ is called the **Spherical Excess** of the triangle ABC and is denoted by the symbol E. It measures the excess of the sum of the angles of a spherical triangle over the sum of the angles of a plane triangle (both being expressed in circular measure) and hence the name.

If we put $2S = A + B + C$, we get

$$S = \tfrac{1}{2}E + \tfrac{1}{2}\pi.$$

Cor. 1. If E_1, E_2 and E_3 be the spherical excesses of the colunar triangles of ABC on the sides a, b and c respectively, then

$$E_1=2A-E,\ E_2=2B-E,\ \text{and}\ E_3=2C-E,$$

and their areas are

$$(2A-E)r^2,\ (2B-E)r^2\ \text{and}\ (2C-E)r^2.$$

Cor. 2. The sum of the areas of any triangle and its colunar triangles is equal to half the area of the sphere.

6.2. Area of a Polygon. Take a polygon of n sides and let Σ denote the sum of its angles. Take any point within the polygon and join it to all the angular points. Then the polygon is divided into n triangles and its area is equal to the sum of the areas of the n triangles. Hence

$$\begin{aligned}\text{area of the polygon} &= (\text{sum of the angles of the } n \text{ triangles} - n\pi)r^2\\ &= (\Sigma+2\pi-n\pi)r^2=\{\Sigma-(n-2)\pi\}r^2\\ &= Er^2,\end{aligned}$$

where E is the spherical excess of the polygon.

Cor. Area of a spherical quadrilateral is

$$(A+B+C+D-2\pi)r^2.$$

6.3. Girard's theorem enables us to get the area of the spherical triangle when the sum of the angles are known. When the three sides or two sides and the

included angle are given, the relations established in the following articles will enable us to find the area.

6.4. Cagnoli's theorem.* *To shew that*

$$\sin \tfrac{1}{2}E = \frac{\sqrt{\{\sin s \sin(s-a) \sin(s-b) \sin(s-c)\}}}{2\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}$$

We have $\sin \tfrac{1}{2}E = \sin (S - \tfrac{1}{2}\pi) = -\cos S$

$$= \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}C - \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}C.$$

Hence substituting the values of $\sin \tfrac{1}{2}(A+B)$ and $\cos \tfrac{1}{2}(A+B)$ from Delambre's analogies (Art. 3.17), we get

$$\sin \tfrac{1}{2}E = \frac{\sin \tfrac{1}{2}C \cos \tfrac{1}{2}C}{\cos \tfrac{1}{2}c} \{\cos \tfrac{1}{2}(a-b) - \cos \tfrac{1}{2}(a+b)\}$$

$$= \frac{\sin C}{\cos \tfrac{1}{2}c} \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b$$

$$= \frac{2n}{\sin a \sin b} \cdot \frac{\sin \tfrac{1}{2}a \sin \tfrac{1}{2}b}{\cos \tfrac{1}{2}c}, \quad \text{by Art. 3.3}$$

$$= \frac{n}{2\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}. \quad \dots \quad (2)$$

6.5. Expressions for $\cos \tfrac{1}{2}E$ and $\tan \tfrac{1}{2}E$. *To shew that*

$$\cos \tfrac{1}{2}E = \frac{1 + \cos a + \cos b + \cos c}{4 \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}$$

and $$\tan \tfrac{1}{2}E = \frac{2n}{1 + \cos a + \cos b + \cos c}.$$

* **Cagnoli**, *Trigonometria*, § 1146. See also **Lexell**, *Acta Petropolitana*, 1782, p. 68. For a geometrical proof see Art. 6.11 below.

We have

$$\cos \tfrac{1}{2}E = \cos (S - \tfrac{1}{2}\pi) = \sin S$$

$$= \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}C + \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}C$$

$$= \{\cos^2 \tfrac{1}{2}C \cos \tfrac{1}{2}(a-b) + \sin^2 \tfrac{1}{2}C \cos \tfrac{1}{2}(a+b)\} \sec \tfrac{1}{2}c$$

by Delambre's analogies Art. 3.17

$$= \{\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b + \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \cos C\} \sec \tfrac{1}{2}c \quad \ldots \quad (3)^*$$

$$= \frac{\cos^2 \tfrac{1}{2}a \cos^2 \tfrac{1}{2}b + \sin \tfrac{1}{2}a \cos \tfrac{1}{2}a \sin \tfrac{1}{2}b \cos \tfrac{1}{2}b \cos C}{\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}$$

$$= \frac{(1+\cos a)(1+\cos b) + \sin a \sin b \cos C}{4 \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}$$

$$= \frac{1+\cos a + \cos b + \cos c}{4 \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}. \quad \ldots \quad \ldots \quad \ldots \quad (4)\dagger$$

Hence dividing (2) by (4), we have

$$\tan \tfrac{1}{2}E = \frac{2n}{1+\cos a + \cos b + \cos c}. \quad \ldots \quad (5)\ddagger$$

* **Lagrange**, *Journal de l'E'cole Polytechnique*, Cahier, 6; **Legendre**, *Géométrie*, Note 10. **Gudermann**, *Niedere Sphärik*, § 152.

† **Euler**, *Acta Petropolitana*, 1778. For a geometrical proof see Art. 6.11 below.

‡ **De Gua**, *Mémoires de l'Académie des Sciences*, Paris, 1783.

6.6. Formulae for Colunar triangles.

Let E_1 be the spherical excess of the colunar triangle $A''BC$. If a_1, b_1, c_1 be the sides and A_1, B_1, C_1 the angles of this triangle, we have

$$E_1 = 2A - E$$

and

$$a_1 = a, \quad b_1 = \pi - b, \quad c_1 = \pi - c.$$

$$A_1 = A, \quad B_1 = \pi - B, \quad C_1 = \pi - C.$$

Also

$$s_1 = \pi - (s-a), \; s_1 - a_1 = \pi - s, \; s_1 - b_1 = s - c$$

and

$$s_1 - c_1 = s - b,$$

so that

$$n_1 = n.$$

Now

$$\sin \tfrac{1}{2}E_1 = \frac{n_1}{2 \cos \tfrac{1}{2}a_1 \cos \tfrac{1}{2}b_1 \cos \tfrac{1}{2}c_1},$$

whence by substituting the values of a_1, b_1, c_1 we have

$$\sin \tfrac{1}{2}E_1 = \sin (A - \tfrac{1}{2}E) = \frac{n}{2 \cos \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c} \quad \ldots(6)$$

Similarly,

$$\sin \tfrac{1}{2}E_2 = \sin (B - \tfrac{1}{2}E) = \frac{n}{2 \sin \tfrac{1}{2}a \cos \tfrac{1}{2}b \sin \tfrac{1}{2}c} \quad \ldots(7)$$

and

$$\sin \tfrac{1}{2}E_3 = \sin (C - \tfrac{1}{2}E) = \frac{n}{2 \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \cos \tfrac{1}{2}c} \quad \ldots(8)$$

Again

$$\cos \tfrac{1}{2}E_1 = \frac{1 + \cos a_1 + \cos b_1 + \cos c_1}{4 \cos \tfrac{1}{2}a_1 \cos \tfrac{1}{2}b_1 \cos \tfrac{1}{2}c_1},$$

whence

$$\cos \tfrac{1}{2}E_1 = \cos (A - \tfrac{1}{2}E) = \frac{1 + \cos a - \cos b - \cos c}{4 \cos \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}, \quad \ldots(9)$$

with similar expressions for $\cos \frac{1}{2}E_2$ and $\cos \frac{1}{2}E_3$.

Also

$$\tan \tfrac{1}{2}E_1 = \tan (A - \tfrac{1}{2}E) = \frac{2n}{1 + \cos a - \cos b - \cos c} \quad \ldots(10)$$

with similar expressions for $\tan \frac{1}{2}E_2$ and $\tan \frac{1}{2}E_3$.

It should be noted here that E_1, E_2 and E_3 being spherical excesses are necessarily positive, and each of them is less than 2π. (Art. 2.9.)

Hence $A - \frac{1}{2}E$, $B - \frac{1}{2}E$, $C - \frac{1}{2}E$ are each less than π.

6.7. L'Huilier's theorem.* *To shew that*

$$\tan \tfrac{1}{4}E = \sqrt{\{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)\}}.$$

We have

$$\tan \tfrac{1}{4}E = \frac{\sin \frac{1}{4}(A + B + C - \pi)}{\cos \frac{1}{4}(A + B + C - \pi)}$$

$$= \frac{\sin \frac{1}{2}(A + B) - \sin \frac{1}{2}(\pi - C)}{\cos \frac{1}{2}(A + B) + \cos \frac{1}{2}(\pi - C)}$$

$$= \frac{\sin \frac{1}{2}(A + B) - \cos \frac{1}{2}C}{\cos \frac{1}{2}(A + B) + \sin \frac{1}{2}C}$$

* See **Legendre** *Géométrie*, Note 10. See also **Grunert's** *Archiv der Math. und Physik.*, XX, 1853, p. 358 for **Gent's** proof of **L'Huilier's** theorem.

$$=\frac{\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b)+\cos\frac{1}{2}c}\cdot\frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}C}$$

by Delambre's analogies,

$$=\frac{\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-a)}{\cos\frac{1}{2}s\cos\frac{1}{2}(s-c)}\left\{\frac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}\right\}^{\frac{1}{2}}$$

by Art. 3.8.

$$=\sqrt{\{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)\}}\quad\ldots(11)$$

6.8. The Lhuilierian.

We have by (11)

$$\tan\tfrac{1}{4}E_1=\sqrt{\{\tan\tfrac{1}{2}s_1\tan\tfrac{1}{2}(s_1-a_1)\tan\tfrac{1}{2}(s_1-b_1)\tan\tfrac{1}{2}(s_1-c_1)\}},$$

whence

$$\tan\tfrac{1}{4}(2A-E)=\sqrt{\{\cot\tfrac{1}{2}s\cot\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)\}}\quad\ldots(12)$$

Similarly,

$$\tan\tfrac{1}{4}(2B-E)=\sqrt{\{\cot\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\cot\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)\}}\quad\ldots(13)$$

and

$$\tan\tfrac{1}{4}(2C-E)=\sqrt{\{\cot\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\cot\tfrac{1}{2}(s-c)\}}\quad\ldots(14)$$

Multiplying together the equations (11), (12), ...(13) and (14) we get

$$\tan\tfrac{1}{4}E\tan\tfrac{1}{4}(2A-E)\tan\tfrac{1}{4}(2B-E)\tan\tfrac{1}{4}(2C-E)$$

$$=\cot\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)=L^2\quad\ldots(15)$$

where L is called the *Lhuilierian* * of the Spherical triangle.

Thus

$$\tan \tfrac{1}{4} E = \frac{L}{\cot \frac{1}{2} s},$$

$$\tan \tfrac{1}{4}(2A - E) = \frac{L}{\tan \frac{1}{2}(s-a)},$$

$$\tan \tfrac{1}{4}(2B - E) = \frac{L}{\tan \frac{1}{2}(s-b)},$$

and

$$\tan \tfrac{1}{4}(2C - E) = \frac{L}{\tan \frac{1}{2}(s-c)}.$$

6.9. Expressions for sin $\frac{1}{4}E$ and cos $\frac{1}{4}E$.

We have

$$\sin^2 \tfrac{1}{4}E = \tfrac{1}{2}(1 - \cos \tfrac{1}{2}E)$$

$$= \tfrac{1}{2}\left\{ 1 - \frac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c} \right\}, \text{ by Art. 6.5}$$

$$= \tfrac{1}{2}\left\{ 1 - \frac{\cos^2 \frac{1}{2}a + \cos^2 \frac{1}{2}b + \cos^2 \frac{1}{2}c - 1}{2 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c} \right\},$$

$$= \frac{1 - \cos^2 \frac{1}{2}a - \cos^2 \frac{1}{2}b - \cos^2 \frac{1}{2}c + 2 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}{4 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}$$

$$= \frac{\sin \frac{1}{2}s \sin \frac{1}{2}(s-a) \sin \frac{1}{2}(s-b) \sin \frac{1}{2}(s-c)}{\cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}. \quad \ldots \quad (16)$$

* The name *Lhuilierian* is suggested by **Dr. Casey** after the name of **L'Huillier** who obtained this expression.

Similarly,

$$\cos^2 \tfrac{1}{4}E = \tfrac{1}{2}(1+\cos \tfrac{1}{2}E)$$

$$= \tfrac{1}{2}\left\{1 + \frac{1+\cos a+\cos b+\cos c}{4\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}\right\}$$

$$= \frac{\cos^2 \tfrac{1}{2}a + \cos^2 \tfrac{1}{2}b + \cos^2 \tfrac{1}{2}c + 2\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c - 1}{4\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}$$

$$= \frac{\cos \tfrac{1}{2}s \cos \tfrac{1}{2}(s-a) \cos \tfrac{1}{2}(s-b) \cos \tfrac{1}{2}(s-c)}{\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c} \quad \dots \quad (17)$$

L'Huilier's theorem is obtained by dividing (16) by (17).

6.10. Expressions for $\sin \frac{1}{4}(2A-E)$ and $\cos \frac{1}{4}(2A-E)$.

Substituting in (16) and (17) the values of the elements of the colunar triangle $A'BC$ from Art. 6.6, we get,

$$\sin^2 \tfrac{1}{4}(2A-E)$$

$$= \frac{\cos \tfrac{1}{2}s \cos \tfrac{1}{2}(s-a) \sin \tfrac{1}{2}(s-b) \sin \tfrac{1}{2}(s-c)}{\cos \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c} \quad \dots \quad (18)$$

and $\cos^2 \frac{1}{4}(2A-E)$

$$= \frac{\sin \tfrac{1}{2}s \sin \tfrac{1}{2}(s-a) \cos \tfrac{1}{2}(s-b) \cos \tfrac{1}{2}(s-c)}{\cos \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c} \quad \dots \quad (19)$$

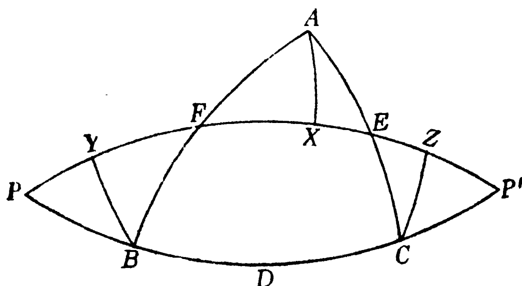
Hence by division, we get

$$\tan^2 \tfrac{1}{4}(2A-E) = \cot \tfrac{1}{2}s \cot \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c),$$

which is the same thing as (12) of Art. 6.8.

6.11. Geometrical representation of the Spherical Excess.

Let D, E and F be the middle points of the sides BC, CA and AB of the triangle ABC, and let EF meet BC produced at P and P'. Then by Art. 5.9, we have



$$P\hat{B}Y = P'\hat{C}Z, \quad F\hat{B}Y = F\hat{A}X \quad \text{and} \quad E\hat{C}Z = E\hat{A}X.$$

Hence $P\hat{B}Y + P'\hat{C}Z = P\hat{B}F + P'\hat{C}E - F\hat{A}X - E\hat{A}X$

$$= 2\pi - (A + B + C) = \pi - E,$$

so that $P\hat{B}Y = P'\hat{C}Z = \frac{1}{2}\pi - \frac{1}{2}E$, *i.e., complement of half of the special excess.*

Now from the right-angled triangle PBY, we have

$$\frac{\sin PBY}{\sin PY} = \frac{1}{\sin PB},$$

but $\qquad PB = \frac{1}{2}\pi - \frac{1}{2}a$ and $PY = \frac{1}{2}\pi - EF$;

therefore

$$\cos \tfrac{1}{2}E = \frac{\cos EF}{\cos \frac{1}{2}a} = \frac{1+\cos a+\cos b+\cos c}{4\cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}.$$

(Ex. 3, p. 39.)

Again $\cos PBY = \sin P \cos PY = \sin P \sin EF$.

But from the triangles PBF and EAF, we have

$$\frac{\sin P}{\sin \frac{1}{2}c} = \frac{\sin F}{\sin PB} \quad \text{and} \quad \frac{\sin EF}{\sin A} = \frac{\sin \frac{1}{2}b}{\sin F}$$

so that $$\cos PBY = \frac{\sin \frac{1}{2}b \sin \frac{1}{2}c \sin A}{\cos \frac{1}{2}a}.$$

Therefore $$\sin \tfrac{1}{2}E = \cos PBY = \frac{n}{2\cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c},$$

which is Cagnoli's formula.

EXAMPLES WORKED OUT

Ex. 1. In a spherical triangle if $\cos C = -\tan \frac{1}{2}a \tan \frac{1}{2}b$,

shew that $C = A + B$.

We have $\cos C = -\tan \frac{1}{2}a \tan \frac{1}{2}b$,

or, $$\cos^2 C = -\frac{\sin \frac{1}{2}a \sin \frac{1}{2}b \cos C}{\cos \frac{1}{2}a \cos \frac{1}{2}b}$$

or, $$\frac{-\cos^2 C}{1-\cos^2 C} = \frac{\sin \frac{1}{2}a \sin \frac{1}{2}b \cos C}{\cos \frac{1}{2}a \cos \frac{1}{2}b + \sin \frac{1}{2}a \sin \frac{1}{2}b \cos C}$$

$$= \tan \tfrac{1}{2}E \cot C, \text{ by Arts. 6.4 and 6.5.}$$

Hence $-\cot C = \tan \frac{1}{2}E = \tan (S - \frac{1}{2}\pi) = -\cot S$,

or, $C = S = \frac{1}{2}(A + B + C)$,

so that $C = A + B$.

Ex. 2. Shew that

$$\sin s = \frac{\{\sin \tfrac{1}{2}E \sin \tfrac{1}{2}(2A-E) \sin \tfrac{1}{2}(2B-E) \sin \tfrac{1}{2}(2C-E)\}^{\frac{1}{2}}}{2 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C}.$$

We have

$$\{\sin \tfrac{1}{2}E \cdot \sin \tfrac{1}{2}(2A-E) \sin \tfrac{1}{2}(2B-E) \sin \tfrac{1}{2}(2C-E)\}^{\frac{1}{2}}$$

$$= \frac{2n^2}{\sin a \sin b \sin c} \text{ by (2) of Art. 6.4 and (6), (7), (8) of Art. 6.6}$$

$$= \frac{2 \sin s \sin (s-a) \sin (s-b) \sin (s-c)}{\sin a \sin b \sin c}, \text{ by Art. 3.9}$$

$$= 2 \sin s \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C, \text{ by Art. 3.8.}$$

Hence the result.

Ex. 3. If E' be the spherical excess of the polar triangle, and E_1, E_2 and E_3 those of the colunar triangles, shew that

$$\tan \tfrac{1}{2}E' = \sqrt{\{\cot \tfrac{1}{2}E \tan \tfrac{1}{2}E_1 \tan \tfrac{1}{2}E_2 \tan \tfrac{1}{2}E_3\}}.$$

(Prouhet.)

Let a', b', c' be the sides and A', B', C' the angles of the polar triangle of ABC, then

$$E' = A' + B' + C' - \pi = 2(\pi - s),$$

$$2s' = a' + b' + c' = 2\pi - E,$$

$$s' - a' = \tfrac{1}{2}(b' + c' - a') = \tfrac{1}{2}(2A - E),$$

$$s' - b' = \tfrac{1}{2}(c' + a' - b') = \tfrac{1}{2}(2B - E),$$

and $$s' - c' = \tfrac{1}{2}(a' + b' - c') = \tfrac{1}{2}(2C - E).$$

Now $$\tan \tfrac{1}{2}E' = \sqrt{\{\tan \tfrac{1}{2}s' \tan \tfrac{1}{2}(s'-a') \tan \tfrac{1}{2}(s'-b') \tan \tfrac{1}{2}(s'-c')\}}$$

by Art. 6.7.

Hence substituting the values, we have

$$\tan \tfrac{1}{2}E' = \tan \tfrac{1}{2}(\pi - s) = \cot \tfrac{1}{2}s$$

$$= \sqrt{\{\tan \tfrac{1}{4}(2\pi - E) \tan \tfrac{1}{4}(2A - E) \tan \tfrac{1}{4}(2B - E) \tan \tfrac{1}{4}(2C - E)\}}$$

$$= \sqrt{\{\cot \tfrac{1}{4}E \tan \tfrac{1}{4}E_1 \tan \tfrac{1}{4}E_2 \tan \tfrac{1}{4}E_3\}}.$$

Examples

If E_1, E_2 and E_3 be the spherical excesses of the colunar triangles on the sides a, b, and c respectively, shew that

1. $$\frac{\sin \frac{1}{2}E}{\sin \frac{1}{2}E_1} = \tan \tfrac{1}{2}b \tan \tfrac{1}{2}c.$$

2. $$\frac{\sin \frac{1}{2}E_1}{\tan \frac{1}{2}a} = \frac{\sin \frac{1}{2}E_2}{\tan \frac{1}{2}b} = \frac{\sin \frac{1}{2}E_3}{\tan \frac{1}{2}c} = \frac{\sin \frac{1}{2}E}{\tan \frac{1}{2}a \tan \frac{1}{2}b \tan \frac{1}{2}c}.$$

3. $$\sin^2 \tfrac{1}{2}E = \frac{\sqrt{\{\sin \frac{1}{2}E \sin \frac{1}{2}E_1 \sin \frac{1}{2}E_2 \sin \frac{1}{4}E_3\}}}{\cot \frac{1}{2}a \cot \frac{1}{4}b \cot \frac{1}{2}c}.$$

4. $\tan \frac{1}{4}E \cot \frac{1}{4}E_1 = \tan \frac{1}{2}s \tan \frac{1}{2}(s-a)$.
   $\tan \frac{1}{4}E \cot \frac{1}{4}E_2 = \tan \frac{1}{2}s \tan \frac{1}{2}(s-b)$.
   $\tan \frac{1}{4}E \cot \frac{1}{4}E_3 = \tan \frac{1}{2}s \tan \frac{1}{2}(s-c)$.

5. $\cot \frac{1}{4}E \tan \frac{1}{4}E_1 \tan \frac{1}{4}E_2 \tan \frac{1}{4}E_3 = \cot^2 \frac{1}{2}s$.
   $\tan \frac{1}{4}E \cot \frac{1}{4}E_1 \tan \frac{1}{4}E_2 \tan \frac{1}{4}E_3 = \tan^2 \frac{1}{2}(s-a)$.
   $\tan \frac{1}{4}E \tan \frac{1}{4}E_1 \cot \frac{1}{4}E_2 \tan \frac{1}{4}E_3 = \tan^2 \frac{1}{2}(s-b)$.
   $\tan \frac{1}{4}E \tan \frac{1}{4}E_1 \tan \frac{1}{4}E_2 \cot \frac{1}{4}E_3 = \tan^2 \frac{1}{2}(s-c)$.

6. In an equilateral triangle of side a, shew that

$$\tan \tfrac{1}{4}E = \tan \tfrac{1}{4}a \sqrt{\tan \tfrac{3}{4}a \tan \tfrac{1}{4}a}.$$

(Dacca Uni., 1930.)

7. In an isosceles triangle shew that

$$\tan \tfrac{1}{4}E = \tan \tfrac{1}{4}c \sqrt{\tan \tfrac{1}{2}(a + \tfrac{1}{2}c) \tan \tfrac{1}{2}(a - \tfrac{1}{2}c)},$$

where a is one of the equal sides.

8. If the angle C of a spherical triangle be a right angle, shew that

(i) $\sin \frac{1}{2}E = \sin \frac{1}{2}a \sin \frac{1}{2}b \sec \frac{1}{2}c$.

(ii) $\cos \frac{1}{2}E = \cos \frac{1}{2}a \cos \frac{1}{2}b \sec \frac{1}{2}c$.

(iii) $\frac{\sin^2 c}{\cos c} \cos E = \frac{\sin^2 a}{\cos a} + \frac{\sin^2 b}{\cos b}$.

9. If the sum of the angles of a spherical triangle be four right angles, shew that

$$\cos^2 \tfrac{1}{2}a + \cos^2 \tfrac{1}{2}b + \cos^2 \tfrac{1}{2}c = 1.$$

10. A given line is divided into two isosceles triangles, and the area of one of them is n times the area of the other ; shew that

$$\tan \tfrac{1}{2}A \cos \theta = \tan \frac{n-1}{n+1} \cdot \frac{A}{2},$$

where A denotes the angle of the lune and θ one of the equal sides.

(*Sci. and Art*, *1894* ; *C.U.*, *M.A.* & *M.Sc.*, *1926.*)

11. Shew that

$$\sin \tfrac{1}{2}E \sin \tfrac{1}{2}E_1 \sin \tfrac{1}{2}E_2 \sin \tfrac{1}{2}E_3 = N^2.$$

12. Shew that

$$\tfrac{1}{2}E = \tan \tfrac{1}{2}a \tan \tfrac{1}{2}b \sin C - \tfrac{1}{2} (\tan \tfrac{1}{2}a \tan \tfrac{1}{2}b)^2 \sin 2C + \ldots\ldots$$

13. If α, β and γ be the arcs joining the middle points of the sides of a spherical triangle, shew that

$$\sin \tfrac{1}{2}E = 2\{\sin \sigma \sin (\sigma - \alpha) \sin (\sigma - \beta) \sin (\sigma - \gamma)\}^{\frac{1}{2}}$$

where $\alpha + \beta + \gamma = 2\sigma$.

14. If the area of a spherical triangle be one-fourth of the area of the sphere, shew that the arcs joining the middle points of its sides are quadrants.

(*London University.*)

15. Shew that

$$\cot \tfrac{1}{2}E = \cot C + \frac{\cot \frac{1}{2}a \cot \frac{1}{2}b}{\sin C}.$$

(*C.U.*, *M.A.* & *M.Sc.*, *1927.*)

APPROXIMATE FORMULAE

6.12. Legendre's Theorem.* *If the sides of a spherical triangle are small compared with the radius of the sphere, then each angle of the spherical triangle exceeds by one third of the spherical excess the corresponding angle of the plane triangle, the sides of which are of the same lengths as the arcs of the spherical triangle.*

Let α, β and γ be the lengths of the arcs forming the sides a, b, c of the spherical triangle ABC, so that the circular measures of the sides are $\frac{\alpha}{r}$, $\frac{\beta}{r}$ and $\frac{\gamma}{r}$, r being the radius of the sphere.

Then

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos\frac{\alpha}{r} - \cos\frac{\beta}{r}\cos\frac{\gamma}{r}}{\sin\frac{\beta}{r}\sin\frac{\gamma}{r}}$$

$$= \frac{\left\{1 - \frac{1}{2!}\frac{\alpha^2}{r^2} + \frac{1}{4!}\frac{\alpha^4}{r^4} - \dots\right\}}{\left\{\frac{\beta}{r} - \frac{1}{3!}\frac{\beta^3}{r^3} + \dots\right\}\left\{\frac{\gamma}{r} - \frac{1}{3!}\frac{\gamma^3}{r^3} + \dots\right\}}$$

$$- \frac{\left\{1 - \frac{1}{2!}\frac{\beta^2}{r^2} + \frac{1}{4!}\frac{\beta^4}{r^4} - \dots\right\}\left\{1 - \frac{1}{2!}\frac{\gamma^2}{r^2} + \frac{1}{4!}\frac{\gamma^4}{r^4} - \dots\right\}}{\left\{\frac{\beta}{r} - \frac{1}{3!}\frac{\beta^3}{r^3} + \dots\right\}\left\{\frac{\gamma}{r} - \frac{1}{3!}\frac{\gamma^3}{r^3} + \dots\right\}}$$

* **Legendre,** *Mémoires de Paris,* 1787, p. 338; *Trigonométrie,* Appendix V. See also **Gauss,** *Disquisitiones generales circa superficies curvas,* §§ 27, 28, and **Mertens,** *Schlömilch's Zeitschrift,* 1875.

Hence neglecting powers of $\frac{1}{r}$ beyond the fourth, we have

$$\cos A = \frac{\frac{1}{2}\cdot\frac{\beta^2+\gamma^2-\alpha^2}{r^2}+\frac{1}{24}\cdot\frac{\alpha^4-\beta^4-\gamma^4-6\beta^2\gamma^2}{r^4}}{\frac{\beta\gamma}{r^2}\left(1-\frac{\beta^2+\gamma^2}{6r^2}\right)}$$

$$=\left\{\frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}+\frac{\alpha^4-\beta^4-\gamma^4-6\beta^2\gamma^2}{24\beta\gamma r^2}\right\}\left\{1+\frac{\beta^2+\gamma^2}{6r^2}\right\}$$

$$=\frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}+\frac{\alpha^4+\beta^4+\gamma^4-2\beta^2\gamma^2-2\gamma^2\alpha^2-2\alpha^2\beta^2}{24\beta\gamma r^2} \quad \ldots \quad (1)$$

If A', B' and C' be the angles of the plane triangle with the sides α, β and γ, we have (Art. 3.6)

$$\cos A' = \frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma},$$

and $\sin^2 A' = 1-\cos^2 A'$

$$=\frac{2\beta^2\gamma^2+2\gamma^2\alpha^2+2\alpha^2\beta^2-\alpha^4-\beta^4-\gamma^4}{4\beta^2\gamma^2}.$$

Hence $\cos A = \cos A' - \frac{\beta\gamma\sin^2 A'}{6r^2}$

$$= \cos A' - \frac{\Delta \sin A'}{3r^2} \quad \ldots \quad \ldots \quad (2)$$

where $\Delta = \frac{1}{2}\beta\gamma \sin A'$, *i.e.*, the area of the plane triangle (Art. 3.10).

Now if θ be the excess of the angle A over the angle A', we have

$$\cos A = \cos (A' + \theta) = \cos A' - \theta \sin A' \text{ approximately,}$$

θ being a very small quantity.

Hence from (2) we have

$$\theta = \frac{\Delta}{3r^2}.$$

Thus $$A = A' + \frac{\Delta}{3r^2}.$$

Similarly, $$B = B' + \frac{\Delta}{3r^2}, \text{ and } C = C' + \frac{\Delta}{3r^2},$$

so that

$$A + B + C = A' + B' + C' + \frac{\Delta}{r^2} = \pi + \frac{\Delta}{r^2},$$

or, $$A + B + C - \pi = \frac{\Delta}{r^2}, \text{ i.e., } E = \frac{\Delta}{r^2}. \quad \ldots \quad (3)$$

Therefore

$$A = A' + \frac{1}{3}E, \quad B = B' + \frac{1}{3}E \text{ and } C = C' + \frac{1}{3}E \ldots . \quad (4)$$

6.13. We have seen in Art. 6.1 that the area of the spherical triangle is Er^2, and from (3) of the previous article we have $Er^2 = \Delta$. Thus *the areas of the spherical triangle and of the plane triangle with sides of the same length are approximately equal, when the sides are very small as compared with the radius of the sphere.*

A closer approximation of the area is given in the following article.

6.14. Approximate value of the spherical excess.*

We have by L'Huilier's theorem (Art. 6.7)

$$\tan \tfrac{1}{4}E = \{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)\}^{\frac{1}{2}}$$

Now

$$\tan \tfrac{1}{2}s = \frac{\frac{1}{2}s - \frac{1}{3!}(\frac{1}{2}s)^3 + \dots}{1 - \frac{1}{2!}(\frac{1}{2}s)^2 + \dots} = \frac{\frac{1}{2}s(1 - \frac{1}{24}s^2 + \dots)}{1 - \frac{1}{8}s^2 + \dots},$$

$$= \tfrac{1}{2}s(1 - \tfrac{1}{24}s^2 + \dots)(1 - \tfrac{1}{8}s^2 + \dots)^{-1} = \tfrac{1}{2}s(1 + \tfrac{1}{12}s^2)$$

approximately.

Hence

$$\tan \tfrac{1}{4}E = \Big[\tfrac{1}{2}s(1 + \tfrac{1}{12}s^2).\tfrac{1}{2}(s-a)\{1 + \tfrac{1}{12}(s-a)^2\}.$$
$$\tfrac{1}{2}(s-b)\{1 + \tfrac{1}{12}(s-b)^2\}.\tfrac{1}{2}(s-c)\{1 + \tfrac{1}{12}(s-c)^2\}\Big]^{\frac{1}{2}},$$
$$= \tfrac{1}{4}\{s(s-a)(s-b)(s-c)\}^{\frac{1}{2}}$$
$$\left\{1 + \frac{s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2}{12} + \dots\right\}^{\frac{1}{2}}$$
$$= \frac{1}{4r^2}\left\{s'(s'-\alpha)(s'-\beta)(s'-\gamma)\right\}^{\frac{1}{2}}$$
$$\left\{1 + \frac{s'^2 + (s'-\alpha)^2 + (s'-\beta)^2 + (s'-\gamma)^2}{12r^2} + \dots\right\}^{\frac{1}{2}}$$

* Gauss, *Disquisitiones*, § 29.

where $2s' = \alpha + \beta + \gamma$.

Thus $\tan \frac{1}{4}E \doteq \dfrac{\Delta}{4r^2}\left\{1 + \dfrac{\alpha^2+\beta^2+\gamma^2}{24r^2}\right\}$

approximately,

or, $$E = \frac{\Delta}{r^2}\left\{1 + \frac{\alpha^2+\beta^2+\gamma^2}{24r^2}\right\}, \quad \ldots \quad (5)$$

since the quantities are very small.

Hence to this order of approximation, the area of the spherical triangle exceeds that of the plane triangle by $\dfrac{1}{24}\,\dfrac{\alpha^2+\beta^2+\gamma^2}{r^2}$ of the latter. If in (5) we neglect the fourth power of r, we get the result (3) of Art. 6.12.

EXAMPLES

1. Shew that a closer approximation for A is given by

$$A = A' + \frac{1}{3}E + \frac{1}{180}\,\frac{E}{r^2}(\beta^2+\gamma^2-2\alpha^2).$$

2. Shew that

$$\frac{\sin A}{\sin B} = \frac{\alpha}{\beta}\left\{1 + \frac{\beta^2-\alpha^2}{6r^2}\left(1 + \frac{7\beta^2-3\alpha^2}{60r^2}\right)\right\}$$

approximately.

3. Shew that for a closer approximation

$$\cos A = \cos A' - \frac{\beta\gamma \sin^2 A'}{6r^2} + \frac{\beta\gamma(\alpha^2-3\beta^2-3\gamma^2)\sin^2 A'}{180r^4}.$$

4. Shew that if $A = A' + \theta$, then approximately

$$\theta = \frac{\beta\gamma \sin A'}{6r^2}\left\{1 + \frac{\alpha^2+7\beta^2+7\gamma^2}{120r^2}\right\}.$$

CHAPTER VII

CIRCLES CONNECTED WITH A GIVEN TRIANGLE

INSCRIBED AND CIRCUMSCRIBED CIRCLES. HART'S CIRCLE.

7.1. Inscribed and Circumscribed Circles. Circles can be described touching the sides of a given spherical triangle or passing through its angular points. The contact again may be internal or external, *i.e.*, the circle may be wholly within the triangle or it may be outside the triangle.

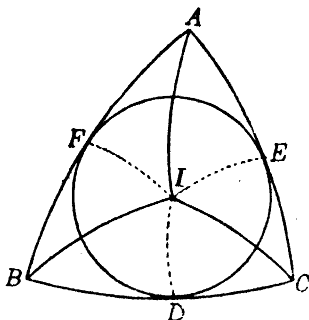
The circle which can be inscribed within the given spherical triangle so as to touch each of its sides internally, is called its *Inscribed Circle* or *Incircle*. Its pole will be the point of intersection of the internal bisectors of the angles of the given triangle. Its angular radius will be denoted by the letter r.

A circle which touches one side of the triangle and the other two sides produced, is called an *Escribed Circle* or *Excircle*. Its pole will be the point of intersection of the bisectors of the external angles. There will be three such excircles to a given triangle, and we denote by the letters r_1, r_2 and r_3 the angular radii of the excircles touching the sides BC, CA and AB respectively. It is evident that the excircle

touching BC is nothing but the incircle of the Colunar triangle $A''BC$. Thus the three excircles are but the incircles of the colunar triangles.

The circle which passes through the angular points of the given triangle, is called its *Circumscribing Circle* or *Circum-circle*. Its pole will be the point of intersection of the arcs bisecting the sides of the triangle at right angles. Its angular radius will be denoted by the letter R.

7.2. The Incircle. *To find the angular radius of the small circle inscribed in a given triangle.*



Let ABC be the given triangle. Bisect the angles B and C by great circular arcs meeting at I. From I draw ID, IE and IF at right angles to the sides.

Then the triangles IBD and IBF having the angles at D and F right angles, the angles at B equal and IB common, are equal in all respects. So also the triangles ICD and ICE are equal. Therefore

$$ID = IE = IF,$$

and the triangles IAE and IAF are equal, so that AI bisects the angle A. Thus the internal bisectors of the angles of the triangle ABC meet at I. A small circle drawn with I as pole and ID as radius will touch the sides at D, E and F and will thus be the incircle of the given triangle.

Now from the triangle IBD, we have by (7) of Art. 4.1

$$\tan ID = \tan \tfrac{1}{2}B \sin BD = \tan \tfrac{1}{2}B \sin (s-b),$$

or denoting ID by r, we have

$$\tan r = \tan \tfrac{1}{2}B \sin (s-b).$$

Similarly, $\tan r = \tan \tfrac{1}{2}A \sin (s-a) = \tan \tfrac{1}{2}C \sin (s-c)$

$$\ldots \quad (1)$$

Again substituting the value of $\tan \tfrac{1}{2}B$ from Art. 3.8 we have

$$\tan r = \sqrt{\frac{\sin (s-a) \sin (s-c)}{\sin s \sin (s-b)}} \sin (s-b) = \frac{n}{\sin s}.$$

$$\ldots \quad (2)$$

Similarly substituting the value of the sines in (1), we get

$$\left.\begin{aligned}\tan r &= \frac{\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C}{\cos \tfrac{1}{2}A} \sin a, \quad \ldots \\ &= \frac{\sin \tfrac{1}{2}C \sin \tfrac{1}{2}A}{\cos \tfrac{1}{2}B} \sin b, \quad \ldots \\ &= \frac{\sin \tfrac{1}{2}A \sin \tfrac{1}{2}B}{\cos \tfrac{1}{2}C} \sin c. \quad \ldots \end{aligned}\right\} \quad \ldots \quad (3)$$

and hence by Arts. 3.13 and 3.14

$$\tan r = \frac{\{-\cos S \cos(S-A) \cos(S-B) \cos(S-C)\}^{\frac{1}{2}}}{2 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.}$$

$$= \frac{N}{2 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.} \quad \ldots \quad (4)^*$$

Again since

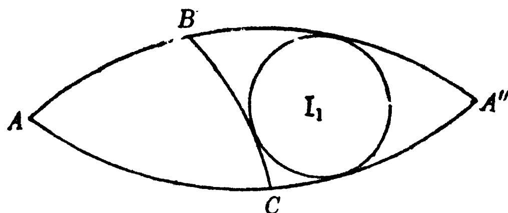
$$\cos S + \cos(S-A) + \cos(S-B) + \cos(S-C) = 4 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$$

we have

$$\cot r = \frac{\{\cos S + \cos(S-A) + \cos(S-B) + \cos(S-C)\}}{2N}. \quad \ldots \quad (5)$$

* Lexell, *Acta Petropolitana*, 1782.

7.3. The Excircle. *To find the angular radii of the escribed circles of the given triangle.*



Let ABC be the given triangle. Produce AB and AC to meet at A''. Then the circle escribed to the side BC is the incircle of the colunar triangle $A''BC$, the parts of which are a, $\pi-b$, $\pi-c$, A, $\pi-B$ and $\pi-C$. If $2s_1$ be the sum of the sides of the colunar triangle, we have

$$s_1=\pi-(s-a),\ s_1-a=\pi-s, \text{ etc.}$$

Hence if r_1 be the radius, we have by Art. 7.2,

$$\tan r_1=\tan \tfrac{1}{2}A \sin (s_1-a)=\tan \tfrac{1}{2}A \sin s. \quad \ldots \quad (6)$$

Proceeding as in Art. 7.2 or substituting the elements of the colunar triangle $A''BC$ in the formulae of Art. 7.2, we get

$$\tan r_1 = \frac{n}{\sin (s-a)}, \quad \ldots \quad \ldots \quad (7)$$

$$= \frac{\cos \tfrac{1}{2}B \cos \tfrac{1}{2}C}{\cos \tfrac{1}{2}A} \sin a, \ldots \quad \ldots \quad (8)$$

$$= \frac{N}{2 \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C}, \quad \ldots \quad (9)$$

and

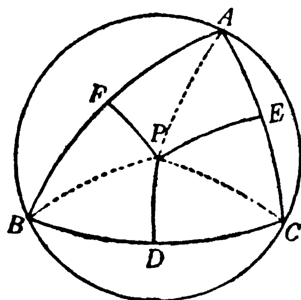
$$\cot r_1 = \frac{\{-\cos S - \cos(S-A) + \cos(S-B) + \cos(S-C)\}}{2N}. \quad \ldots \quad (10)$$

7.4. The radii r_2 and r_3 of the other two ex-circles are easily obtained in the same manner or by appropriate interchange of letters in r_1. Thus

$$\tan r_2 = \tan \tfrac{1}{2}B \sin s = \frac{n}{\sin(s-b)}, \text{ etc.,}$$

and $$\tan r_3 = \tan \tfrac{1}{2}C \sin s = \frac{n}{\sin(s-c)}, \text{ etc.}$$

7.5. The Circumcircle. *To find the angular radius of the small circle described about a given triangle.*



Let ABC be the given triangle. Bisect the sides BC and CA at right angles at D and E by great circular arcs meeting at P. Join PA, PB and PC.

Then the triangles PBD and PCD, having $BD = CD$, PD common and the angles at D right angles, are equal in all respects, so that $PB = PC$. Similarly from the equality of the triangles PCE and PAE, we have $PC = PA$, so that $PA = PB = PC$.

Hence a circle with P as pole and radius PA will pass through the angular points of ABC, and will thus be the circumcircle of the triangle.

Now from the triangle BPD, we have by Art. 4.1

$$\tan BD = \tan BP \cos PBD = \tan BP \cos (S-A),$$

or denoting the radius by R, we have

$$\tan \tfrac{1}{2}a = \tan R \cos (S-A),$$

i.e.,
$$\tan R = \frac{\tan \frac{1}{2}a}{\cos (S-A)}. \quad \ldots$$

Similarly,

$$\tan R = \frac{\tan \frac{1}{2}b}{\cos (S-B)} = \frac{\tan \frac{1}{2}c}{\cos (S-C)}. \quad \ldots \qquad \ldots \quad (11)$$

Substituting the value of $\tan \frac{1}{2}a$ from Art. 3.13 we have

$$\tan R = \left\{ \frac{-\cos S}{\cos (S-A) \cos (S-B) \cos (S-C)} \right\}^{\frac{1}{2}}$$

$$= -\frac{\cos S}{N}. \quad \ldots \qquad \ldots \quad (12)$$

Again since (Ex. 11, p. 56)

$$\cos(S-A) = -\cos S \cot \tfrac{1}{2}b \cdot \cot \tfrac{1}{2}c,$$

we have $$\tan R = -\frac{\tan \frac{1}{2}a \tan \frac{1}{2}b \tan \frac{1}{2}c}{\cos S}. \quad \ldots \quad (13)$$

Also $$-\cos S = \frac{n}{2\cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}, \text{ (Ex. 15, p. 56)}$$

Hence $$\tan R = \frac{2\sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c}{n}. \quad \ldots \quad (14)\,*$$

We have from Ex. 14, p. 56.

$$\frac{\cos(S-A)}{\sin A} = \frac{\cos \frac{1}{2}b \cos \frac{1}{2}c}{\cos \frac{1}{2}a},$$

so that $$\tan R = \frac{\sin \frac{1}{2}a}{\sin A \cos \frac{1}{2}b \cos \frac{1}{2}c}. \quad \ldots \quad (15)$$

Again since

$$\sin(s-a) + \sin(s-b) + \sin(s-c) - \sin s$$
$$= 4 \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c,$$

we have

$$\tan R = \frac{1}{2n}\left\{\sin(s-a) + \sin(s-b) + \sin(s-c) - \sin s\right\}. \quad \ldots \quad (16)$$

* Lexell, *l.c.* This result follows at once from Ex. 16, p. 56.

7.6. Circumcircles of the colunar triangles. *To find the angular radii of the circumcircles of the three colunar triangles.*

Let R_1, R_2 and R_3 be the angular radii of the circumcircles of the colunar triangles on the sides a, b and c respectively. The elements of the triangle $A''BC$ are a, $\pi-b$, $\pi-c$, A, $\pi-B$ and $\pi-C$. Hence substituting these values in the formulae of Art. 7.5, we get the formulae for R_1, the circumradius of the colunar triangle $A''BC$.

Thus $$\tan R_1 = -\frac{\tan \frac{1}{2}a}{\cos S} \quad \dots \quad (17)$$

$$= \frac{\cos (S-A)}{N} \quad \dots \quad (18)$$

$$= \frac{\tan \frac{1}{2}a \cot \frac{1}{2}b \cot \frac{1}{2}c}{\cos (S-A)} \quad \dots \quad (19)$$

$$= \frac{2 \sin \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c}{n} \quad \dots \quad (20)$$

$$= \frac{\sin \frac{1}{2}a}{\sin A \sin \frac{1}{2}b \sin \frac{1}{2}c} \quad \dots \quad (21)$$

$$= \frac{1}{2n} \left\{ \sin s - \sin(s-a) + \sin(s-b) + \sin(s-c) \right\}. \quad \dots \quad (22)$$

Similarly, $$\tan R_2 = -\frac{\tan \frac{1}{2}b}{\cos S} = \frac{\cos (S-B)}{N}, \text{ etc.,}$$

and $\tan R_3 = -\dfrac{\tan \frac{1}{2}c}{\cos S} = \dfrac{\cos (S-C)}{N}$, etc.

7.7. Inscribed and Circumscribed circles of the Polar triangle.

Let $A'B'C'$ be the polar triangle of ABC. Now I, the incentre of ABC, is equidistant from its three sides and hence equidistant from their poles A', B' and C' (Ex. 6, p. 12). Hence

$$IA' = IB' = IC' = \tfrac{1}{2}\pi - r,$$

i.e., a circle with I as pole and IA' as radius will pass through B' and C'. Thus,

The pole of the incircle of any triangle is also the pole of the circumcircle of the polar triangle, and the radius of the incircle of the triangle is equal to the complement of the circumradius of the polar triangle.

Similar reasoning applies to the case of excircles also. Thus the poles of the excircles are the same as the poles of circumcircles of the respective colunar triangles of the polar triangle and the radii of the former are the complements of the respective circumradii of the later.

Again since ABC is also the polar triangle of $A'B'C'$, we have the supplemental relation,

The pole of the circumcircle of any triangle is also the pole the incircle of the polar triangle and the circumradius of the triangle is equal to the complement of the radius of the incircle of the polar triangle.

It follows from the above that if the radius of the incircle of a triangle is known, the radius of the circumcircle of the polar triangle as also of the given triangle is at once obtained.

EXAMPLES WORKED OUT

Ex. **1.** Shew that $(\cot r + \tan R)^2$

$$= \frac{1}{4n^2}(\sin a + \sin b + \sin c)^2 - 1$$

$$= \frac{1}{4N^2}(\sin A + \sin B + \sin C)^2 - 1.$$

We have from Arts. 7.2 and 7.5

$$\cot r + \tan R = \frac{\sin s}{n} + \frac{1}{2n}\left\{\sin(s-a) + \sin(s-b) + \sin(s-c) - \sin s\right\}$$

$$= \frac{1}{2n}\left\{\sin s + \sin(s-a) + \sin(s-b) + \sin(s-c)\right\}$$

$$= \frac{1}{n}\left\{\sin \tfrac{1}{2}(b+c)\cos \tfrac{1}{2}a + \sin \tfrac{1}{2}a \cos \tfrac{1}{2}(b-c)\right\}.$$

Hence squaring both sides, we have

$$(\cot r + \tan R)^2 = \frac{1}{n^2}\left\{\sin^2 \tfrac{1}{2}(b+c)\cos^2 \tfrac{1}{2}a + \sin^2 \tfrac{1}{2}a \cos^2 \tfrac{1}{2}(b-c) + 2\sin \tfrac{1}{2}a \cos \tfrac{1}{2}a \sin \tfrac{1}{2}(b+c)\cos \tfrac{1}{2}(b-c)\right\};$$

$$= \frac{1}{4n^2} \Big\{ \{1 - \cos (b+c)\} (1 + \cos a)$$

$$+ (1 - \cos a)\{1 + \cos (b-c)\} + 2 \sin a (\sin b + \sin c) \Big\}$$

$$= \frac{1}{2n^2} \Big\{ 1 + \sin a \sin b + \sin b \sin c + \sin c \sin a$$

$$- \cos a \cos b \cos c \Big\}$$

$$= \frac{1}{\ } \Big\{ (\sin a + \sin b + \sin c)^2 - (1 - \cos^2 a$$

$$- \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c) \Big\}$$

$$= \frac{1}{4n^2} (\sin a + \sin b + \sin c)^2 - 1.$$

Again since $\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C} = \dfrac{n}{N}$, (Ex. 7, p. 55)

we have

$$(\cot r + \tan R)^2 = \frac{1}{4N^2} (\sin A + \sin B + \sin C)^2 - 1.$$

Similarly, $(\cot r_1 - \tan R)^2 = \dfrac{1}{4n^2} (\sin b + \sin c - \sin a)^2 - 1,$

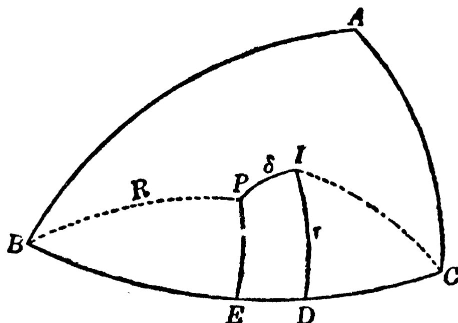
$$(\cot r_2 - \tan R)^2 = \frac{1}{4n^2} (\sin c + \sin a - \sin b)^2 - 1,$$

and $(\cot r_3 - \tan R)^2 = \dfrac{1}{4n^2} (\sin a + \sin b - \sin c)^2 - 1.$

Ex. 2. **Angular distance between the poles of the circumcircle and the incircle.**

. If δ be the length of the great circular arc joining the poles of the incircle and the circumcircle of a triangle, then will

$$\cos^2 \delta = \sin^2 r \cos^2 R + \cos^2 (R - r).$$



Let I and P be the poles of the incircle and circumcircle of the triangle ABC, and let PI be denoted by δ. Through I and P draw two secondaries to BC meeting it at D and E respectively. Then we have by Art. 3.7

$$\cos \delta = \sin ID \sin PE + \cos ID \cos PE \cos ED.$$

But $\qquad BD = s - b,\ BE = \frac{1}{2}a$; hence $ED = \frac{1}{2}(c - b)$.

Also $\qquad ID = r,\ \sin PE = \sin R \sin PBE = \sin R \sin (S - A)$,

and $\qquad \cos PE = \dfrac{\cos R}{\cos \frac{1}{2}a}$.

Hence $\qquad \cos \delta = \sin r \sin R \sin (S - A) + \cos r \cos R \dfrac{\cos \frac{1}{2}(c - b)}{\cos \frac{1}{2}a}$

$$=\sin r \sin R \sin (S-A)+\cos r \cos R \frac{\sin \frac{1}{2}(B+C)}{\cos \frac{1}{2}A},$$

by Delambre's first analogy (Art. 3.17)

$$=\sin r \cos R \left\{ \tan R \sin (S-A)+\cot r \frac{\sin \frac{1}{2}(B+C)}{\cos \frac{1}{2}A} \right\}$$

$$=\sin r \cos R \left\{ \frac{-\cos S \sin (S-A)+2 \cos \frac{1}{2}B \cos \frac{1}{2}C \sin \frac{1}{2}(B+C)}{N} \right\},$$

by Arts. 7.2 and 7.5

$$=\sin r \cos R \left\{ \frac{\sin A+\sin B+\sin C}{2N} \right\}.$$

Therefore we have by Ex. 1,

$$\cos^2 \delta=\sin^2 r \cos^2 R\{(\cot r+\tan R)^2+1\}$$

$$=\sin^2 r \cos^2 R+\cos^2(R-r).$$

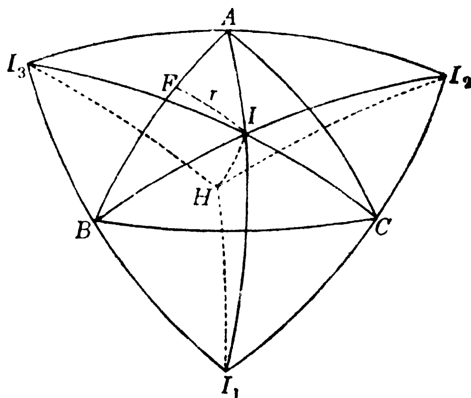
7.8. Hart's Circle. In the plane geometry we have the well-known theorem of Feuerbach that the inscribed and escribed circles of a plane triangle are all touched by another circle, namely, the Ninepoints Circle. Sir Andrew Hart discovered in 1861 * that the theorem holds in the case of spherical triangles also. He demonstrated that the inscribed circles of a spherical triangle and its colunar triangles are all

* See *Quarterly Journal of Mathematics*, Vol. IV, p. 260.

touched by another small circle. This circle touches internally the incircle of the triangle and externally the incircles of the colunar triangles.

7.9. Spherical Radius of Hart's Circle.

Let ABC be the given triangle, and r, r_1, r_2, r_3 the radii and I, I_1, I_2, I_3 the poles of the inscribed and escribed circles. Let ρ be the radius and H the centre of Hart's circle. Then since Hart's circle has internal contact with the incircle and external contact with the excircles of ABC, we have



$$HI=\rho-r,\ HI_1=\rho+r_1,\ HI_2=\rho+r_2 \text{ and } HI_3=\rho+r_3.$$

Now since the angle A is bisected internally by AI and externally by AI_3, they are at right angles to

each other. Thus AI_1 is an altitude of the triangle $I_1I_2I_3$. Similarly BI_2 and CI_3 are the other altitudes.

Let $2v$, $2v_1$, $2v_2$ and $2v_3$ be the sines of the triangles $I_1I_2I_3$, II_2I_3, II_3I_1 and II_1I_2, then

$$2v=\sin I_2I_3 \sin AI_1,\ 2v_2=\sin I_3I_1 \sin BI,$$

$$2v_1=\sin I_2I_3 \sin AI,\ 2v_3=\sin I_1I_2 \sin CI.$$

If IF be drawn perpendicular on AB, we have $IF=r$, and

$$\sin r=\sin AI \sin \tfrac{1}{2}A \text{ and } \sin r_1=\sin AI_1 \sin \tfrac{1}{2}A,$$

so that $\sin r : \sin r_1=\sin AI : \sin AI_1$.

Hence $v : v_1= \dfrac{1}{\sin r} : \dfrac{1}{\sin r_1}$

Similarly

$$v : v_1 : v_2 : v_3=\frac{1}{\sin r} : \frac{1}{\sin r_1} : \frac{1}{\sin r_2} : \frac{1}{\sin r_3}.$$

Applying Dr. Casey's Theorem (Art. 5.10) on the triangle $I_1I_2I_3$ we have

$$v_1 \cos HI_1+v_2 \cos HI_2+v_3 \cos HI_3=v \cos HI,$$

or,

$$\frac{\cos (\rho+r_1)}{\sin r_1}+\frac{\cos (\rho+r_2)}{\sin r_2}+\frac{\cos (\rho+r_3)}{\sin r_3}=\frac{\cos (\rho-r)}{\sin r}.$$

i.e., $\cos\rho\,(\cot r_1+\cot r_2+\cot r_3)-3\sin\rho$

$=\cos\rho\cot r+\sin\rho.$

Thus $\qquad 4\tan\rho=\cot r_1+\cot r_2+\cot r_3-\cot r$

$$=\frac{1}{n}\left\{\sin(s-a)+\sin(s-b)+\sin(s-c)-\sin s\right\}$$

$=2\tan R,$

where R is the circumradius of the triangle ABC.

Hence $\qquad \tan\rho=\frac{1}{2}\tan R.$

7.10. Angular distance of the pole of Hart's circle from the vertices of the given triangle.

The lengths of the arcs joining H to A, B and C can be obtained with the help of Art 5.1. Thus applying the theorem to the arc I_3AI_2 we have

$$\cos HI_3\sin AI_2+\cos HI_2\sin AI_3=\cos AH\sin I_2I_3,$$

or, $\qquad \cos(\rho+r_3)\sin AI_2+\cos(\rho+r_2)\sin AI_3$

$$=\cos AH\sin(AI_2+AI_3).$$

But $\qquad \sin AI_2=\dfrac{\sin r_2}{\cos\frac{1}{2}A},\quad \sin AI_3=\dfrac{\sin r_3}{\cos\frac{1}{2}A}.$

$$\cos AI_2=\cos r_2\cos(s-c),\ \cos AI_3=\cos r_3\cos(s-b).$$

Hence substituting these values in the above equality, we have

$$\sin r_2 \cos r_3 + \cos r_2 \sin r_3 - 2 \tan \rho \sin r_2 \sin r_3$$

$$= \frac{\cos AH}{\cos \rho} \left\{ \sin r_2 \cos r_3 \cos (s-b) + \sin r_3 \cos r_2 \cos (s-c) \right\},$$

or, $\cot r_2 + \cot r_3 - 2 \tan \rho$

$$= \frac{\cos AH}{\cos \rho} \left\{ \cot r_3 \cos (s-b) + \cot r_2 \cos (s-c) \right\},$$

whence substituting the values of $\cot r_2$, $\cot r_3$ from Art. 7.4, and of $\tan \rho$ from Art. 7.9 we get

$$\sin (s-b) + \sin (s-c) - 2 \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c$$

$$= \frac{\cos AH}{\cos \rho} \left\{ \sin (s-c) \cos (s-b) + \sin (s-b) \cos (s-c) \right\},$$

Hence simplifying, we have

$$\cos AH = \frac{\cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}{\cos \tfrac{1}{2}a} \cos \rho.$$

Similarly $\cos BH = \dfrac{\cos \tfrac{1}{2}c \cos \tfrac{1}{2}a}{\cos \tfrac{1}{2}b} \cos \rho,$

and $$\cos CH = \frac{\cos \frac{1}{2}a \cos \frac{1}{2}b}{\cos \frac{1}{2}c} \cos \rho.$$

7.11. The lengths AH, BH and CH can be obtained easily without previous knowledge of the value of ρ. Thus from the previous article we have

$$\cot r_2 + \cot r_3 - 2 \tan \rho$$
$$= \frac{\cos AH}{\cos \rho} \left\{ \cot r_3 \cos (s-b) + \cot r_2 \cos (s-c) \right\}$$

And applying Art. 5·1 to the arc AI_1, we have

$$\cos (\rho - r) \sin AI_1 - \cos (\rho + r_1) \sin AI$$
$$= \cos AH \sin (AI_1 - AI),$$

which on simplification becomes

$$\cot r_1 - \cot r - 2 \tan \rho$$
$$= -\frac{\cos AH}{\cos \rho} \left\{ \cot r \cos (s-a) - \cot r_1 \cos s \right\},$$

Thus $$\cot r_2 + \cot r_3 - 2 \tan \rho = \frac{\cos AH \sin a}{n \cos \rho},$$

and $$\cot r_1 - \cot r - 2 \tan \rho = -\frac{\cos AH \sin a}{n \cos \rho}.$$

Hence equating we have

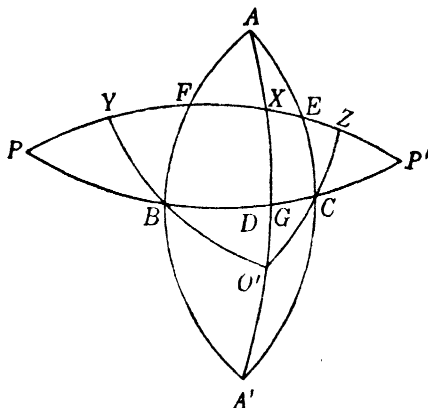
$$\tan \rho = \tfrac{1}{4} (\cot r_1 + \cot r_2 + \cot r_3 - \cot r) = \tfrac{1}{2} \tan R.$$

$$\text{and } \cos AH = \tfrac{1}{2}\frac{n\cos\rho}{\sin a}\left\{\cot r_2 + \cot r_3 - \cot r_1 + \cot r\right\}$$

$$= \frac{\cos\frac{1}{2}b\cos\frac{1}{2}c}{\cos\frac{1}{2}a}\cos\rho.$$

Thus the value of ρ is simultaneously obtained with that of AH.

7.12. Baltzer's Theorem.* *The pole of the great circle through the middle points of two sides of a triangle is also the pole of the circumcircle of the colunar triangle.*



Draw AX, BY and CZ at right angles to the great circle EF passing through the middle point E and F

* **Baltzer**, *Trigonometrie*, § 5.

of the sides AC and AB of the triangle ABC. Let these perpendiculars meet at O'. Then O' is the pole of the great circle EF.

We have by Art. 5.9 $\quad AY = BY = CZ = p$ (say),

then $\quad O'A = \frac{1}{2}\pi + p$ and $O'B = O'C = \frac{1}{2}\pi - p$.

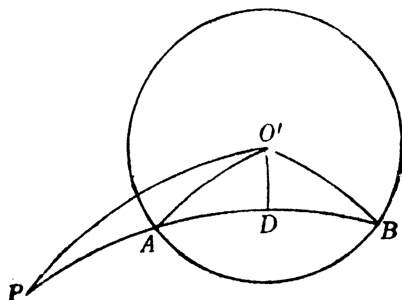
Hence $\quad O'B = O'C = \frac{1}{2}\pi - (O'A - \frac{1}{2}\pi) = \pi - O'A = O'A'$,

where A' is the point diametrically opposite to A.

Thus the point O' is equidistant from the points B, C and A', *i.e.*, the vertices of the colunar triangle $A'BC$ and hence is the pole of its circumcircle.

7.13. **Theorem.*** *If from a fixed point P on the surface of a sphere, a great circular arc be drawn to cut a given small circle in A and B, then will*

$$\tan \tfrac{1}{2}PA \tan \tfrac{1}{2}PB = \text{constant}.$$



* Lexell, *Acta Petropolitana*, 1782, p. 65.

Let O' be the pole of the given small circle. Draw $O'D$ perpendicular to AB. Then the triangles $O'AD$ and $O'BD$ are symmetrically equal and hence $AD=BD$.

Now from the triangle $PO'D$, we have

$$\cos PO'=\cos PD \cos O'D,$$

and from the triangle $AO'D$, we have

$$\cos AO'=\cos AD \cos O'D.$$

Hence $$\frac{\cos PO'}{\cos AO'}=\frac{\cos PD}{\cos AD},$$

or, $$\frac{\cos AD-\cos PD}{\cos AD+\cos PD}=\frac{\cos AO'-\cos PO'}{\cos AO'+\cos PO'},$$

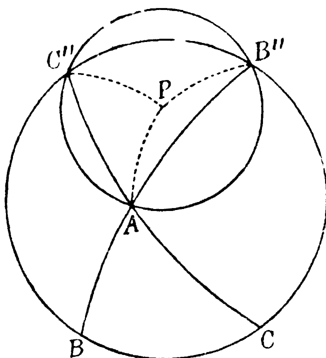
i.e., $$\tan \tfrac{1}{2}(PD-AD) \tan \tfrac{1}{2}(PD+AD)$$
$$=\tan \tfrac{1}{2}(PO'-AO') \tan \tfrac{1}{2}(PO'+AO').$$

Thus $$\tan \tfrac{1}{2}PA \tan \tfrac{1}{2}PB=\tan \tfrac{1}{2}(\delta-\rho) \tan \tfrac{1}{2}(\delta+\rho)$$
$$=\text{constant},$$

where δ denotes the angular distance PO' and ρ the angular radius AO'.

It is evident that this result does not depend on the positions of A and B, so that it holds for all positions of the arc PAB drawn through P. The constant $\tan \frac{1}{2}(\delta-\rho) \tan \frac{1}{2}(\delta+\rho)$ is called *the spherical power of the point P with respect to the circle. It is positive when P is outside the circle, negative when P is inside.*

7.14. Lexell's locus.* *The base and the area of a spherical triangle being given, the locus of the vertex is a small circle.*



Let BC be the given base, and B'' and C'' be the points diametrically opposite to B and C respectively. Then in the triangle $AB''C''$, the angle $B''=\pi-B$ and $C''=\pi-C$. Suppose P to be the pole of the circum-circle of the triangle $AB''C''$. Join PA, PB'' and PC''.

Then we have

$$P\hat{B}''C''=P\hat{C}''B'',\ P\hat{C}''A=P\hat{A}C'' \text{ and } P\hat{A}B''=P\hat{B}''A.$$

Therefore $B''+C''-A=P\hat{B}''C''+P\hat{C}''B''=$

$$2\,P\hat{B}''C''=2\,P\hat{C}''B''.$$

* **Lexell,** ***Acta Peteopolitana,*** **1781, I, p. 112.**

Hence if the angle $PB''C''$ or $PC''B''$ is known, the pole P can be determined.

Now the area of the triangle ABC is given; hence its spherical excess E is also known. But

$$E = A + B + C - \pi = A + \pi - B'' + \pi - C'' - \pi$$
$$= \pi - (B'' + C'' - A).$$

Thus $B'' + C'' - A$, *i.e.*, the angle $PB''C''$ or $PC''B''$ is known, so that P is determined and the circum-circle of $AB''C''$ is completely known.

As A is a variable point, it follows that the locus of A is a small circle through B'' and C''—the circum-circle of the triangle $AB''C''$.

Examples.

Prove the following relations for a spherical triangle :—

1. $\tan r \tan r_1 \tan r_2 \tan r_3 = n^2$.
   $\cot r \tan r_1 \tan r_2 \tan r_3 = \sin^2 s$.
   $\tan r \cot r_1 \tan r_2 \tan r_3 = \sin^2 (s-a)$.
   $\tan r \tan r_1 \cot r_2 \tan r_3 = \sin^2 (s-b)$.
   $\tan r \tan r_1 \tan r_2 \cot r_3 = \sin^2 (s-c)$.

2. $\cot R \cot R_1 \cot R_2 \cot R_3 = N^2$.
   $\tan R \cot R_1 \cot R_2 \cot R_3 = \cos^2 S$.
   $\cot R \tan R_1 \cot R_2 \cot R_3 = \cos^2 (S-A)$.
   $\cot R \cot R_1 \tan R_2 \cot R_3 = \cos^2 (S-B)$.
   $\cot R \cot R_1 \cot R_2 \tan R_3 = \cos^2 (S-C)$.

3. $\cot r_1 : \cot r_2 : \cot r_3 : \cot r$
   $= \sin (s-a) : \sin (s-b) : \sin (s-c) : \sin s$.
   $\tan R_1 : \tan R_2 : \tan R_3 = \cos (S-A) : \cos (S-B) : \cos (S-C)$.

4. $\cot r_1 + \cot r_2 + \cot r_3 - \cot r = 2 \tan R.$

$\tan R_1 + \tan R_2 + \tan R_3 - \tan R = 2 \cot r.$

$\cot r - \cot r_1 + \cot r_2 + \cot r_3 = 2 \tan R_1.$

$\tan R - \tan R_1 + \tan R_2 + \tan R_3 = 2 \cot r_1.$

5. $\cot r \sin s = \cot \frac{1}{2}A \cot \frac{1}{2}B \cot \frac{1}{2}C.$

6. $\tan R + \cot r = \tan R_1 + \cot r_1 = \tan R_2 + \cot r_2$

$$= \tan R_3 + \cot r_3 = \tfrac{1}{2}(\cot r + \cot r_1 + \cot r_2 + \cot r_3).$$

7. $\tan R \tan R_1 + \tan R_2 \tan R_3 = \cot r \cot r_1 + \cot r_2 \cot r_3.$

8. $$\frac{\tan r_1 + \tan r_2 + \tan r_3 - \tan r}{\cot r_1 + \cot r_2 + \cot r_3 - \cot r} = \tfrac{1}{2}(1 + \cos a + \cos b + \cos c).$$

9. $$\frac{\tan^2 R + \tan^2 R_1 + \tan^2 R_2 + \tan^2 R_3}{\cot^2 r + \cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3} = 1.$$

10. $$\frac{\tan^2 R + \tan^2 R_1 - \tan^2 R_2 - \tan^2 R_3}{\cot^2 r + \cot^2 r_1 - \cot^2 r_2 - \cot^2 r_3} = -\frac{\cos A}{\cos a}.$$

11. $$\frac{\tan r}{\tan R} = \frac{\cos (S-A) \cos (S-B) \cos (S-C)}{2 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C}.$$

12. $\operatorname{cosec}^2 r = \cot (s-b) \cot (s-c) + \cot (s-c) \cot (s-a)$

$$+ \cot (s-a) \cot (s-b).$$

$\operatorname{cosec}^2 r_1 = \cot (s-b) \cot (s-c) - \cot s \cot (s-b) - \cot s \cot (s-c).$

13. $$\frac{\cot (s-a)}{\sin^2 r_1} + \frac{\cot (s-b)}{\sin^2 r_2} + \frac{\cot (s-c)}{\sin^2 r_3} + \frac{2 \cot s}{\sin^2 r}$$

$$= 3 \cot (s-a) \cot (s-b) \cot (s-c).$$

14. $\operatorname{cosec}^2 r_1 + \operatorname{cosec}^2 r_2 + \operatorname{cosec}^2 r_3 - \operatorname{cosec}^2 r$

$$= -2 \cot s \{\cot (s-a) + \cot (s-b) + \cot (s-c)\}.$$

15. $\sqrt{1 + (\cot r_1 - \tan R)^2} + \sqrt{1 + (\cot r_2 - \tan R)^2}$

$$+ \sqrt{1 + (\cot r_3 - \tan R)^2} = \sqrt{1 + (\cot r + \tan R)^2}.$$

16. Shew that in an equilateral spherical triangle

$$\tan R = 2 \tan r.$$

17. ABC is an equilateral spherical triangle, P the pole of the circle circumscribing it, and Q any point on the sphere : shew that

$$\cos QA + \cos QB + \cos QC = 3 \cos PA \cos PQ.$$

(*C. U. M. A. & M. Sc., 1926.*)

18. If δ be the angular distance between the poles of the circumcircle and the incircle of a spherical triangle, shew that

$$\frac{\cos \delta}{\sin r \sin R} = \frac{\sin a + \sin b + \sin c}{4 \sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c},$$

and

$$\sec^2 R \sec^2 r \sin^2 \delta = \tan^2 R - 2 \tan R \tan r.$$

(*London Univ. Exam. Papers.*)

19. If δ_1, δ_2 and δ_3 denote the angular distances between the poles of the circumcircle and excircles of a spherical triangle, shew that

$$\cos^2 \delta_1 = \cos^2 R \sin^2 r_1 + \cos^2 (R + r_1).$$

$$\cos^2 \delta_2 = \cos^2 R \sin^2 r_2 + \cos^2 (R + r_2).$$

$$\cos^2 \delta_3 = \cos^2 R \sin^2 r_3 + \cos^2 (R + r_3).$$

$$\sin^2 \delta_1 = \sin^2 (R + r_1) - \cos^2 R \sin^2 r_1.$$

$$\sin^2 \delta_2 = \sin^2 (R + r_2) - \cos^2 R \sin^2 r_2.$$

$$\sin^2 \delta_3 = \sin^2 (R + r_3) - \cos^2 R \sin^2 r_3.$$

20. If I, I_1, I_2 and I_3 denote the poles of the inscribed and escribed circles of a spherical triangle, shew that

$$\cos II_1 : \cos II_2 : \cos II_3 = \frac{\cos r_1}{\cos (s-a)} : \frac{\cos r_2}{\cos (s-b)} : \frac{\cos r_3}{\cos (s-c)}.$$

21. If S, S_1, S_2 and S_3 denote the sums of the angles of a spherical triangle and its three colunars, shew that

$$S + S_1 + S_2 + S_3 = 3\pi.$$

22. If P, P_1, P_2 and P_3 denote the poles of the circumscribed circles of a spherical triangle and its three colunars, shew that

$$\tan PP_1 : \tan PP_2 : \tan PP_3$$
$$= \cos \tfrac{1}{2}a \sin (S-A) : \cos \tfrac{1}{2}b \sin (S-B) : \cos \tfrac{1}{2}c \sin (S-C).$$

23. If in a spherical triangle, the vertical angle be equal to the sum of the base angles, then the pole of the circumcircle will lie in the base.

24. If ABC be a spherical triangle having each side a quadrant, I the pole of the incircle, P any point on the sphere, then will

$$(\cos PA + \cos PB + \cos PC)^2 = 3 \cos^2 PI.$$

25. Two circles whose radii are $\cot^{-1} \alpha$ and $\cot^{-1} \beta$ touch externally. Shew that the angle between their common tangents is

$$2 \cos^{-1} \frac{2\sqrt{\alpha\beta - 1}}{\alpha + \beta}.$$

(*C. U. M. A. & M. Sc.*, *1928.*)

26. PAB is a spherical triangle, of which the side AB is fixed, and the angles PAB and PBA are supplementary. Prove that the vertex P lies on a fixed great circle.

(*Science and Art, 1899.*)

27. Two circles of angular radii, α and β, intersect orthogonally on a sphere of radius r; find in any manner the area common to the two.

(*London University.*)

28. If H be the centre of Hart's circle for the spherical triangle ABC, shew that

$$\cos AH : \cos BH : \cos CH = \sec^2 \tfrac{1}{2}a : \sec^2 \tfrac{1}{2}b : \sec^2 \tfrac{1}{2}c.$$

29. If t_1, t_2 and t_3 be the lengths of the tangents from the vertices A, B and C to Hart's circle, shew that

$$\cos t_1 = \sec \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c.$$

$$\cos t_2 = \cos \tfrac{1}{2}a \sec \tfrac{1}{2}b \cos \tfrac{1}{2}c.$$

$$\cos t_3 = \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \sec \tfrac{1}{2}c.$$

30. If the side AB of the spherical triangle ABC be intersected by Hart's circle at points distant λ and μ from A, shew that

$$\tan \tfrac{1}{2}\lambda = \frac{\cos \tfrac{1}{2}a - \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}{\cos \tfrac{1}{2}b \sin \tfrac{1}{2}c}$$

and
$$\tan \tfrac{1}{2}\mu = \frac{\cos \tfrac{1}{2}b \sin \tfrac{1}{2}c}{\cos \tfrac{1}{2}a + \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c}.$$

31. Shew that the intercept made by Hart's circle on the side AB is given by

$$2 \tan^{-1} \left\{ \frac{\cos^2 \tfrac{1}{2}a - \cos^2 \tfrac{1}{2}b}{2 \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \sin \tfrac{1}{2}c} \right\}.$$

32. Shew that the angle between Hart's circle and a side of the triangle is equal to the difference of the angles of the triangle adjacent to that side.

33. $ABCD$ is a spherical quadrilateral inscribed in a small circle, and the diagonals AC and BD intersect at P: shew that

$$\tan \tfrac{1}{2}PA \tan \tfrac{1}{2}PC = \tan \tfrac{1}{2}PB \tan \tfrac{1}{2}PD.$$

34. ABC is a spherical triangle, and a small circle cuts BC in P and P', CA in Q and Q', AB in R and R' : shew that

$$\frac{\sin AQ \sin AQ'}{\cos^2 \frac{1}{2}QQ'} = \frac{\sin AR \sin AR'}{\cos^2 \frac{1}{2}RR'}$$

and

$$\frac{\sin BP \sin BP'}{\sin CP \sin CP'} \cdot \frac{\sin CQ \sin CQ'}{\sin AQ \sin AQ'} \cdot \frac{\sin AR \sin AR'}{\sin BR \sin BR'} = 1.$$

35. P is the pole of the circumcircle of the spherical triangle ABC, and AP is produced to meet BC in D; shew that if δ denotes PD,

$$\tan \tfrac{1}{2}BPD \tan \tfrac{1}{2}CPD = \frac{\sin (R-\delta)}{\sin (R+\delta)}.$$

If the angle A be a right angle, shew that

$$\cos^2 R = \frac{\sin (R-\delta)}{\sin (R+\delta)}.$$
